PANJER CLASS UNITED
One formula for the probabilities of the Poisson, Binomial, and Negative Binomial distribution

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Abstract. This paper gives a formula representing all discrete loss distributions of the Panjer class (Poisson, Binomial, and Negative Binomial) in one. Further it provides an overview of the many Negative Binomial variants used by actuaries.

Keywords. Panjer class, (a,b,0) class, discrete loss distribution, Negative Binomial

Resumen. En este artículo se presenta una fórmula conjunta para las probabilidades de las distribuciones de la clase Panjer (Poisson, binomial, binomial negativa). Además se da una mirada general a las variantes de la distribución binomial negativa utilizadas por los actuarios.

Palabras clave. Clase de Panjer, clase (a,b,0), distribución del número de siniestros, distribución binomial negativa

1. Introduction

The three well-known discrete loss distributions Poisson, Binomial, and Negative Binomial are closely related. First of all, they form the Panjer (a,b,0) class (see Panjer and Willmot, 1992, section 7.2; Klugman *et al.*, 2004, appendix B.2). Secondly, the Poisson distribution is a limiting case of the two other distributions, which finally have their origin in the modelling of Bernoulli trials. The traditional representations of the probability (mass) functions of the three distributions look quite different, though.

In this paper we add to the unified view of these distributions by presenting a common formula for the probabilities, being both instructive and convenient for practical implementation.

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Section 2 states classical parametrisations of the three distributions, adding a Binomial variant being less common but easier to compare to the other members of the (a,b,0) class. Section 3 provides a formula representing the three distributions altogether.

To give some orientation in view of the confusing variety of parametrisations used in the actuarial world (especially for Negative Binomial) section 4 collects and classifies the variants most frequently found in the actuarial literature.

2. Representations of the Panjer distributions

In order to see how the above distributions of loss numbers N are related we shall first summarize: probability function (pf) $p_k = P(X=k)$, probability generating function (pgf) $E(z^N)$, expected value E(N), and dispersion D(N) = Var(N)/E(N) of the three distributions.

Among the – not few – different parametrisations that can be found for the Panjer distributions in the literature (for a discussion see section 4) we are particularly interested in those using the expected value as a parameter, in the following denoted by $\lambda>0$. (We leave the degenerate case $\lambda=0$ aside, where all distributions coincide.) Note that in general insurance one often deals with models having $\lambda = v\theta$ where v is a measure of the size of the risk (or portfolio of risks) and θ is the loss frequency per volume unit (see e.g. Mack, 1999, section 1.4.2; Bühlmann and Gisler, 2005, section 4.10). For simplicity we will, however, always write λ.

- The Poisson distribution essentially has one common representation (P), using the expectation as the (only) parameter.
- For the Binomial distribution we state the classical parametrisation (B1) using as parameters the number of trials m and the probability of success p. We add another one (B2) where p is replaced by the expected value (of successes) $\lambda = mp$. $p < 1$ means $\lambda < m$.
- For the Negative Binomial distribution we first state the classical parametrisation (NB1) coming from Bernoulli trials, using as parameters the number of successes α (originally integer-valued but extendable to all positive real numbers) and the probability of success $p<1$. Then (NB2) we replace p by the expectation (of failures) $\lambda = \alpha(1-p)/p$.

See the following Table:

Table 1. Short overview of the Panjer (a,b,0) class

	pf	pgf	E(N)	D(N)
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	λ	1
B1	$\binom{m}{k}p^k(1-p)^{m-k}$	$(1-p+pz)^m$	mp	$1-p$
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	λ	$1-\frac{\lambda}{m}$
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	$\frac{\alpha(1-p)}{p}$	$\frac{1}{p}$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	λ	$1+\frac{\lambda}{\alpha}$

It seems that the traditional representations, namely B1 and NB1, somehow obscure the relationship between the three distributions. If we instead look at B2 and NB2 at least the pgfs look very similar, and here and in the formulae for the dispersion there is an obvious correspondence between α and m, or merely –m. This well-known correspondence (see e.g. Heckman and Meyers, 1983, sections 3 and 5) will turn out to be the key of the common representation.

3. The all-in-one formula

Proposition: The formula

$$p_k = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}, \qquad k = 0, 1, 2, \ldots \qquad (1)$$

describes the probability (mass) function of all distributions of the Panjer (a,b,0) class. The parameter λ is the expected value, which can take on all positive real numbers. The parameter α can take on the following values:

a) $\alpha \in]0; \infty[$: Negative Binomial.
b) $\alpha = \infty$, $\alpha = -\infty$: Poisson. (1) is well defined in this infinite case as the limits exist and coincide.
c) $\alpha \in]-\infty; -\lambda[\cap \mathbf{Z}$: Binomial. The parameter α here is restricted to integers $-m$ satisfying $m > \lambda$.

The corresponding probability generating function is given by

$E(z^N) = \left(1 - \frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$, which again is well defined for infinite α.

Definition 1: We call the above parametrisation of the (a,b,0) distributions **Panjer United (PanU)**.

Proof: First we convert (1) into a known pf for each of the three cases.

a) We only have to rearrange the terms of NB2, noting that $\binom{\alpha+k-1}{k} =$

$$= \frac{1}{k!}\prod_{i=0}^{k-1}(\alpha+i)\,, \quad \left(\frac{\alpha}{\alpha+\lambda}\right)^{\alpha} = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}, \quad \left(\frac{\lambda}{\alpha+\lambda}\right)^{k} = \lambda^k \prod_{i=0}^{k-1}\frac{1}{\alpha+\lambda}.$$

b) Recall that lim $(1+y/\alpha)^{\alpha} = e^y$ for $\alpha \to \infty$ and $\alpha \to -\infty$, therefore the first factor in (1) equals $e^{-\lambda}$. Since the third factor equals 1 we are done.

c) If we set $m := -\alpha$ in B2 we get

$$\binom{m}{k}\frac{\lambda^k (m-\lambda)^{m-k}}{m^m} = \frac{1}{k!}\,\frac{\lambda^k(-\alpha-\lambda)^{-\alpha-k}}{(-\alpha)^{-\alpha}}\prod_{i=0}^{k-1}(-\alpha-i) =$$

$$= \frac{\lambda^k}{k!}\,(-1)^k\,\frac{\alpha^{\alpha}}{(\alpha+\lambda)^{\alpha+k}}\,(-1)^k\prod_{i=0}^{k-1}(\alpha+i) = \left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$$

Note that (1) is well defined and valid even for k>m. In this case the p_k equal zero as the products $\prod_{i=0}^{k-1}(\alpha+i)$ contain the factor α+m=0.

The pgf formula is obvious for finite α, and for infinite α the reasoning is as in b) with $y = \lambda(z-1)$.

Finally, to see that we have a one-to-one correspondence of the parameters appearing in the usual representations of the (a,b,0) class and in the PanU formula we only have to check that the restrictions for negative α coincide: The Binomial distribution has a positive integer m being greater than λ. This translates to a negative integer α and to the condition –α > λ, being exactly case c) of the Proposition.

Remark 1: Formula (1) splits the probabilitiy p_k in three components: p_0, a term completing the Poisson formula, and finally a product describing in a way the deviation from Poisson. The latter product shows at a glance the well-known fact that it is not possible to extend the parameter space to any further negative values for α: Its first factor $\frac{\alpha+0}{\alpha+\lambda}$ must be positive, otherwise p_0 and p_1 would have different sign. Hence the denominator must be negative, i.e. –α > λ. Now assume that α is not an integer. Then all factors $\frac{\alpha+k}{\alpha+\lambda}$, and with them all p_k, are non-zero. Thus the factors must be postive, otherwise p_k and p_{k+1} would have different sign. Hence all numerators $\alpha+k$ must be negative, but this is impossible as k is unlimited.

Corollary 1: In the above parametrisation the Panjer recursion reads

$p_k = p_{k-1}(a + b/k)$ with $a = \frac{\lambda}{\alpha+\lambda}$, $b = \frac{(\alpha-1)\lambda}{\alpha+\lambda}$ and we have

$$E(N) = \lambda, \quad Var(N) = \lambda\left(1+\frac{\lambda}{\alpha}\right), \quad CV^2(N) = \frac{1}{\lambda}+\frac{1}{\alpha}, \quad D(N) = 1+\frac{\lambda}{\alpha}.$$

Again all formulae are well defined for infinite α.

Proof: From (1) we immediately get

$$p_k = p_{k-1}\frac{\lambda}{k}\frac{\alpha+k-1}{\alpha+\lambda} = p_{k-1}\left(\frac{\lambda}{\alpha+\lambda}+\frac{(\alpha-1)\lambda}{(\alpha+\lambda)k}\right)$$

The following formulae are well-known consequences of the Panjer recursion (see Klugman *et al*., 2004, appendix B.2):

$$E(N) = \frac{a+b}{1-a}, \quad Var(N) = \frac{a+b}{(1-a)^2}, \quad CV^2(N) = \frac{1}{a+b}, \quad D(N) = \frac{1}{1-a}.$$

Plugging in $a+b = \frac{\alpha\lambda}{\alpha+\lambda}$, $1-a = \frac{\alpha}{\alpha+\lambda}$ yields the claimed results.

Corollary 2: In the PanU representation the n-th derivative of the probability generation function equals $\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha-n}\lambda^n\prod_{i=0}^{n-1}\left(1+\frac{i}{\alpha}\right)$, which is again well defined for the whole PanU parameter space.

Proof: From the PanU pgf formula $\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$ we get the claimed result for finite α in a straightforward manner via induction. (Note that the derivative formula is correct for all integers n>0, even in the Binomial case where for n > m = –α it equals zero.) As for the infinite case, the limit of the derivative formula for $\alpha \to \pm\infty$ equals $\lambda^n e^{\lambda(z-1)}$, which is exactly the n-th derivative of the limit of the pgf formula.

Corollary 3: All moments of the (a,b,0) distributions can be written as linear combinations of the terms $\lambda^n\prod_{i=0}^{n-1}\left(1+\frac{i}{\alpha}\right)$, n = 1, 2, 3, …, being well defined for the whole PanU parameter space.

Proof: The n-th factorial moment of a discrete loss distribution equals the n-th derivative of the pgf, evaluated at z=1 (see Panjer and Willmot, 1992, section 2.4). In the PanU representation this value is given by the above term. As all moments are linear combinations of the factorial moments, we are done.

Remark 2: Corollary 3 makes clear that representations for higher moments coming originally from the Negative Binomial representation NB2 are extendable to the whole (a,b,0) class. E.g. we can state without any further calculation that the well-known Negative Binomial skewness formula

$$E((N-E(N))^3) = \lambda\left(1+\frac{\lambda}{\alpha}\right)\left(1+\frac{2\lambda}{\alpha}\right)$$

is not only valid for positive α (NB2) but for the whole PanU parameter space, including the cases of negative skewness for Binomial distributions with $-\alpha/2 < \lambda < -\alpha$, i.e. $0.5 < p < 1$.

Now we define a Panjer United variant without infinite parameter values by replacing the parameter α by its inverse c = 1/α. This parameter was named “contagion” (see e.g. Heckman and Meyers, 1983, sections 3 and 5, see Panjer and Willmot, 1992, sections 3.6, 6.9, and 6.11 for the description of the underlying stochastic processes) in order to give an intuitive meaning to deviations from the Poisson distribution: A higher / lower probability of loss after the occurrence of a loss can be interpreted as positive / negative contagion of losses. Here $0 < c < \infty$ is the Negative Binomial case (positive contagion), c = 0 corresponds to Poisson (no contagion) and negative c is the Binomial case (negative contagion) having the quite intricate parameter restriction $c = -1/m$ with integer $m > \lambda > 0$. This parameter space is complex, however, maybe a bit less complex than the classical representation of the Panjer class in terms of a and b (see Panjer and Willmot, 1992, section 6.6). Furthermore the parameters λ and c, describing expectation and contagion, are very intuitive and, last but not least, enable the practitioner to implement the three distributions in a single procedure, e.g. for the purpose of simulation.

Proposition 2 / Definition 2: The formula

PanU* $$p_k = (1+c\lambda)^{-1/c}\,\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{1+ci}{1+c\lambda}, \qquad k = 0, 1, 2, \ldots \quad (2)$$

describes the probability (mass) function of all distributions of the Panjer (a,b,0) class. The parameter λ is the expected value, which can take on all positive real numbers. The parameter c can take on the following values:

a) $c \in \,]0;\infty[$: Negative Binomial.

b) $c = 0$: Poisson. (2) is well defined as the limit of the first factor exists.

c) $c \in \,]-1/\lambda; 0[\, \cap \{1/z \mid z \in \mathbf{Z}^*\}$: Binomial.

The corresponding pgf is given by $E(z^N) = (1-c\lambda(z-1))^{-1/c}$, having the n-th derivative $(1-c\lambda(z-1))^{-1/c-n}\,\lambda^n \prod_{i=0}^{n-1}(1+ic)$, all being well defined on the whole parameter space. All moments can be written as linear combinations of the terms $\lambda^n \prod_{i=0}^{n-1}(1+ic)$.

The coefficients of the Panjer recursion are $a = \frac{c\lambda}{1+c\lambda}$, $b = \frac{(1-c)\lambda}{1+c\lambda}$

and we have $E(N) = \lambda$, $Var(N) = \lambda + c\lambda^2$, $CV^2(N) = \frac{1}{\lambda} + c$, $D(N) = 1 + c\lambda$, and $E((N-E(N))^3) = \lambda(1+c\lambda)(1+2c\lambda)$.

Conclusion: The "united" representation of the three Panjer (a,b,0) distributions via a common probability function is both convenient for practical implementation and instructive as it makes clearer how closely related and at the same time how different the three distributions are: Binomial and Negative Binomial appear very similar in the PanU/PanU* representation but they are in a way the opposite sides of a coin, being connected, or rather separated, by the limiting case Poisson.

4. The Negative Binomial world

In order to give an overview we enhance Table 1 by adding several variants of the Negative Binomial distribution, all being useful in certain areas but partly tricky to convert into each other. We start from the table provided by Mack (1999, section 1.4.2) showing essentially three different ways of interpreting the distribution, all using α but having different second parameters:

- Bernoulli trial with probability p: NB1
- Expectation λ: NB2 (see also Johnson *et al*., (2005, section 5.1) who dedicate their whole chapter 5 to the Negative Binomial distribution)
- Poisson-Gamma: If the parameter of a Poisson distribution is Gamma distributed with density $\beta^{\alpha}\, x^{\alpha-1}e^{-\beta x}\big/\Gamma(\alpha)$ then the mixed distribution is a variant NB3 inheriting the parameters α and β from the Gamma distribution (see also e.g. Bühlmann and Gisler, 2005, section 2.4).

However, actuaries found two more useful second parameters, closely related to (and easy to confuse with) NB1 and NB3, respectively:

- There is a close variant NB1b of the Bernoulli trial using the complementary probability q=1–p (see e.g. Johnson *et al.*, 2005, section 5.1).
  As q equals the Panjer recursion parameter a this can also be seen as a representation with this parameter of the Panjer recursion. The latter interpretation is by the way extendable to the Binomial case (see Schröter (1990, section 4, Proposition 1) showing this for an extension of the (a,b,0) class).
- There is a Poisson-Gamma variant NB4 using ξ=1/β according to an alternative definition of the Gamma density having parameters α and the inverse of β (see Klugman *et al*., 2004, section 4.6.3).
  The same representation comes about from a totally different approach (see Johnson *et al.*, 2005, section 5.1) – by applying the generalized

binomial theorem (for real-valued exponents) to $1 = \left((1+\xi)-\xi\right)^{-\alpha} =$
$= \sum_{k=0}^{\infty}\binom{-\alpha}{k}(1+\xi)^{-\alpha-k}(-\xi)^k = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\frac{\xi^k}{(1+\xi)^{\alpha+k}}$.

Remark: In this paper we have restricted ourselves to parametrisations using α or the inverse c. For completeness we mention two further representations (see Johnson *et al.*, 2005, section 5.1) combining the expectation with one of the above second parameters:

- NB2/1b: λ together with q yields the pgf $\left(\frac{1-q}{1-qz}\right)^{\lambda\left(\frac{1}{q}-1\right)}$
- NB2/4: λ together with ξ yields the pgf $\left(1-\xi(z-1)\right)^{-\lambda/\xi}$

The conversion of the parameters is as follows:

$$p = 1-q = \frac{\alpha}{\alpha+\lambda} = \frac{\beta}{1+\beta} = \frac{1}{1+\xi},$$
$$q = 1-p = \frac{\lambda}{\alpha+\lambda} = \frac{1}{1+\beta} = \frac{\xi}{1+\xi},$$
$$\lambda = \frac{\alpha(1-p)}{p} = \frac{\alpha q}{1-q} = \frac{\alpha}{\beta} = \alpha\xi,$$
$$\beta = \frac{p}{1-p} = \frac{1}{q}-1 = \frac{\alpha}{\lambda} = \frac{1}{\xi},$$
$$\xi = \frac{1}{p}-1 = \frac{q}{1-q} = \frac{\lambda}{\alpha} = \frac{1}{\beta}.$$

Table 2 shows 10 distributions (1 Poisson, 2 Binomial, 5 Negative Binomial, and the 2 all-in-one representations) providing for each: probability function, probability generating function, probability of no losses, expectation, variance, squared coefficient of variation, dispersion, and at last the parameters a and b of the Panjer recursion.

The table makes clear that for any of these Negative Binomial representations there is something it describes better (in a simpler way) than the other variants do – but in contrast it has more intricate formulae for other quantities that could be of interest. It seems that there is no "best" parametrisation for all actuarial needs, which arguably is why so many different ones have been established. However, NB2, apart from the possible extension to the whole (a,b,0) class shown in this paper, has further advantages: It involves the expected value λ, being in practice the quantity of main interest (being indeed often seen as even more important than the specification of the most adequate model). The second parameter α shows how much the distribution deviates from the popular Poisson model. Finally (see Panjer and Willmot, 1992, section 9.8) the Maximum Likelihood estimators for these two parameters are independent, which is not the case for some other parametrisations.

As the coexistence of so many – partly very similar – parametrisations is a persistent source of errors and misunderstandings, it might possibly be a good idea to agree, at least for educational purposes, on a standard among actuaries, e.g. consistent names for the variants.

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Table 2. Thorough overview of the Panjer (a,b,0) class

	pf	**pgf**	**P(N = 0)**
P	$\frac{\lambda^k}{k!}e^{-\lambda}$	$e^{\lambda(z-1)}$	$e^{-\lambda}$
B1	$\binom{m}{k}p^k(1-p)^{m-k}$	$(1-p+pz)^m$	$(1-p)^m$
B2	$\binom{m}{k}\frac{\lambda^k(m-\lambda)^{m-k}}{m^m}$	$\left(1+\frac{\lambda}{m}(z-1)\right)^m$	$\left(1-\frac{\lambda}{m}\right)^m$
NB1	$\binom{\alpha+k-1}{k}p^\alpha(1-p)^k$	$\left(\frac{1-(1-p)z}{p}\right)^{-\alpha}$	p^α
NB1b	$\binom{\alpha+k-1}{k}(1-q)^\alpha q^k$	$\left(\frac{1-q}{1-qz}\right)^\alpha$	$(1-q)^\alpha$
NB2	$\binom{\alpha+k-1}{k}\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha\left(\frac{\lambda}{\alpha+\lambda}\right)^k$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(\frac{\alpha}{\alpha+\lambda}\right)^\alpha$
NB3	$\binom{\alpha+k-1}{k}\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}$	$\left(1-\frac{z-1}{\beta}\right)^{-\alpha}$	$\left(\frac{\beta}{1+\beta}\right)^\alpha$
NB4	$\binom{\alpha+k-1}{k}\frac{\xi^k}{(1+\xi)^{\alpha+k}}$	$(1-\xi(z-1))^{-\alpha}$	$(1+\xi)^{-\alpha}$
PanU	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{\alpha+i}{\alpha+\lambda}$	$\left(1-\frac{\lambda}{\alpha}(z-1)\right)^{-\alpha}$	$\left(1+\frac{\lambda}{\alpha}\right)^{-\alpha}$
PanU*	$(1+c\lambda)^{-1/c}\frac{\lambda^k}{k!}\prod_{i=0}^{k-1}\frac{1+ci}{1+c\lambda}$	$(1-c\lambda(z-1))^{-1/c}$	$(1+c\lambda)^{-1/c}$

p, q $\in$]0;1[; m positive integer; λ, α, β, ξ, c $\in$]0; ∞[plus further values for α, c in PanU

E(N)	**Var(N)**	**CV²(N)**	**D(N)**	**a**	**b**	
λ	λ	$\frac{1}{\lambda}$	1	0	λ	**P**
mp	$mp(1-p)$	$\frac{1-p}{mp}$	$1-p$	$\frac{-p}{1-p}$	$\frac{(m+1)p}{1-p}$	**B1**
λ	$\lambda\left(1-\frac{\lambda}{m}\right)$	$\frac{1}{\lambda}-\frac{1}{m}$	$1-\frac{\lambda}{m}$	$\frac{-\lambda}{m-\lambda}$	$\frac{(m+1)\lambda}{m-\lambda}$	**B2**
$\frac{\alpha(1-p)}{p}$	$\frac{\alpha(1-p)}{p^2}$	$\frac{1}{\alpha(1-p)}$	$\frac{1}{p}$	$1-p$	$(\alpha-1)(1-p)$	**NB1**
$\frac{\alpha q}{1-q}$	$\frac{\alpha q}{(1-q)^2}$	$\frac{1}{\alpha q}$	$\frac{1}{1-q}$	q	$(\alpha-1)q$	**NB1b**
λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**NB2**
$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta}\left(1+\frac{1}{\beta}\right)$	$\frac{1+\beta}{\alpha}$	$1+\frac{1}{\beta}$	$\frac{1}{1+\beta}$	$\frac{\alpha-1}{1+\beta}$	**NB3**
$\alpha\xi$	$\alpha\xi(1+\xi)$	$\frac{1+\xi}{\alpha\xi}$	$1+\xi$	$\frac{\xi}{1+\xi}$	$\frac{(\alpha-1)\xi}{1+\xi}$	**NB4**
λ	$\lambda\left(1+\frac{\lambda}{\alpha}\right)$	$\frac{1}{\lambda}+\frac{1}{\alpha}$	$1+\frac{\lambda}{\alpha}$	$\frac{\lambda}{\alpha+\lambda}$	$\frac{(\alpha-1)\lambda}{\alpha+\lambda}$	**PanU**
λ	$\lambda+c\lambda^2$	$\frac{1}{\lambda}+c$	$1+c\lambda$	$\frac{c\lambda}{1+c\lambda}$	$\frac{(1-c)\lambda}{1+c\lambda}$	**PanU***