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# Dynamic Equilibrium and the Structure of Premiums in a Reinsurance Market

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*Abstract*

In this paper we present an economic equilibrium analysis of a reinsurance market. The continuous-time model contains the principal components of uncertainty: about the time instants at which accidents take place, and about claim sizes given that accidents have occurred.

We give sufficient conditions on preferences for a general equilibrium to exist, with a Pareto optimal allocation, and derive the premium functional via a representative agent pricing theory. The marginal utility process of the reinsurance market is represented by the density process for random measures, which opens up for numerous applications to premium calculations, some of which are presented in the last section.

The Hamilton-Jacobi-Bellman equations of individual dynamic optimization are established for proportional treaties, and the term structure of interest rates is found in this reinsurance syndicate.

The paper attempts to reach a synthesis between the classical actuarial risk theory of insurance, in which virtually no economic reasoning takes place but where the net reserve is represented by a stochastic process, and the theory of equilibrium price formation at the heart of the economics of uncertainty.

**Key words:** Reinsurance, Exchange Equilibrium, Intertemporal Economic Model, Market Marginal Utility Process, Densities for Stochastic Processes, Random Measure, Marked Point Processes, Dynamic Optimization, Term Structure of Interest Rates, Incomplete Models, Non-Proportional Treaties.

## 1. Introduction

We present an economic equilibrium model for a reinsurance market, in which there exists uncertainty about the claim sizes in the market and uncertainty regarding the time points of occurrence of accidents. In this infinite dimensional setting we first present a set of sufficient conditions guaranteeing the existence of an economic equilibrium with a Pareto optimal allocation. A certain representation property of martingales is available, and any risk can be decomposed into a proportional and a non-proportional component treaty. The market's marginal utility process is derived using the Saddle Point Theorem. The term structure of short-time borrowing is found in the present model, and the premium functional is established, using the density process for random measures. The Hamilton-Jacobi-Bellman equations for marked point processes are established for the individual, dynamic optimal proportional components of the reinsurance strategies.

Finally we give applications of the general theory to premium computations in reinsurance.

One of the most important results in this paper is related to an interpretation of the marginal disutility process in the market, resulting in a splitting of the price of risk into two components: One related to the claim sizes in the market, and the other to the attitude towards frequency risk at each time instant. Given the diversity of lines of reinsurance in the real world, our results can roughly be interpreted as follows: In some lines the uncertainty about frequency appears to be far more important than the uncertainty about conditional claim sizes, given that accidents have occurred (e.g. auto insurance). In other lines it is the other way around, like in oil or marine reinsurance, whereas in yet other lines both sources of uncertainty may be judged to be of importance, and thus ought to be incorporated in a proper analysis. In Section 7 we return to some examples. Another important feature is the classification of any general risky contract into two components: one corresponding to a proportional treaty, and the other to a nonproportional contract. These two components are orthogonal, in a manner to be made precise in the paper, where the proportional treaty is as close as possible in some sense to the general risk we started out with. Again "closeness" will be precisely defined in the paper. Dynamic programming is used to find the optimal proportional component of a reinsurance treaty. Also the term structure of interest rates is derived within the jump framework of the present paper, and several new aspects emerge from this model, distinguishing it from the analogous continuous type models.

In our model for a pure exchange economy we assume that short-term borrowing is possible, and the associated interest rate we ultimately determine endogenously. The vector of the portfolios of the I insurers, constituting the market, is denoted by $\mathbf{x}(t) = \{x_1(t), x_2(t), \cdots, x_I(t)\}$, $t \in T = [0,T]$. At time zero $\mathbf{x}(0)$ equals the initial reserves of the insurers held at the beginning of the period, and also represents the contracts that the insurers have negotiated at this time. After the reinsurance treaties have been settled, insurer i's portfolio process, which we shall sometimes call insurer i's cash flow or net reserve, is denoted by $y_i(t)$, $t \in T$, $i \in I = \{1, 2, \cdots, I\}$. Uncertainty is modeled by a filtered probability space $(\boldsymbol{\Omega}, F, \{F_t\}, P)$ where the "usual" regularity conditions are satisfied, on which the vector processes $\{\mathbf{x}(t), t \in T\}$ and $\{\mathbf{y}(t), t \in T\}$ are both defined. The filtration $\{F_t\}$ is right-continuous, and $F_t \supseteq F_s$, $t > s$, $F = F_T$. The set of subsets $F_t$ represents all the events that could possibly be observed at time t by all the reinsurers. We assume that information is generated by the process $\mathbf{x}$, i.e.

$$F_t = \sigma\{\mathbf{x}(s), s \leq t\}, \qquad t \in T, \tag{1.1}$$

so that the information the insurers are assumed to base their reinsurance strategies on at each time instant is solely based on past and present information related to the net reserves originally given in the market. The insurers are assumed to have homogeneous beliefs represented by the probability measure P. In a reinsurance market this is considered to be a reasonable assumption, since trade is supposed to take place under conditions of *umberrimae fidei*, and no information is presumed to be hidden. As the vector of initial net reserves $\mathbf{x}(t)$ evolves over time, the corresponding vector of actually held net reserves after reinsurance treaties have taken place, $\mathbf{y}(t)$, can be continually renegotiated as uncertainty is revealed bit by bit, in accordance with the preferences of the insurers, subject to the budget constraints, and depending on past and present values of $\mathbf{x}(t)$, and $\mathbf{y}(t)$ (i.e. depending upon $F_t$). The set of possible outcomes in the world is denoted by $\boldsymbol{\Omega}$, with generic element $\omega$. Thus $y_i(t,\omega)$ = the present net reserve at time t of insurer i if $\omega \in \boldsymbol{\Omega}$ is the state of the world. Throughout we make the square integrability assumption

$$\mathbf{x}(T) \in L^2(\boldsymbol{\Omega}, F, P). \tag{1.2}$$

The paper is organized as follows: In Section 2 we present the economic model, also containing the stochastic dynamics governing the process $\{\mathbf{x}(t), t \in T\}$, and we give sufficient conditions for a static equilibrium to exist. In Section 3 we derive the market marginal utility process and discuss proportional and non-proportional treaties in relation to the presented model. Here we also explain what we mean by a *stochastic equilibrium*. In Section 4 we indicate how optimal proportional dynamic reinsurance strategies can be derived using stochastic control theory in the case where a stochastic equilibrium exists. In Section 5 we establish the existence of a stochastic equilibrium for the reinsurance economy under non-proportional spanning, by using known existence results from the corresponding static model. In Section 6 we develop the term structure of the interest rates in the present insurance syndicate. In Section 7 we present applications to premium calculations in the reinsurance market, and in Section 8 we offer some concluding remarks.



## 2. The economic model

### 2.1. *Introduction*

In this section we describe the primitives for a stochastic reinsurance exchange economy; a model for uncertainty and revelation of information over time, a collection of stochastic processes representing the insurance risks in the market, endowments and initial net reserves, and preferences. The existence of a static equilibrium is demonstrated in this section, first under weak conditions on preferences, later under more restrictive assumptions yielding stronger results.

### 2.2. *The net reserves*

We assume that the net reserves $\{x_i(t), t \in T\}$ of insurer i can be decomposed as follows

$$x_i(t) = a_i(t) - z_i(t), \qquad i \in I, \qquad t \in T. \tag{2.1}$$

Here $a_i(t)$ equals the assets of insurer i, which means the initial reserves plus accumulated premiums in [0, t], and $z_i(t)$ equals the total claims payments under insurer i's contracts paid up to time t. The accumulated premiums must be determined endogenously, and we return to this issue later on. Consider the two quantities

$$\mathbf{V}^+(t,\mathbf{x}) = \int_{[0,t)} d\mathbf{x}(s) \tag{2.2}$$

and

$$\mathbf{V}^-(t,\mathbf{x}) = -\int_{[t,T]} d\mathbf{x}(s). \tag{2.3}$$

At each time t the vector $\mathbf{V}^+(t,\mathbf{x})$ is the value of past premiums less claims payments of all the insurers in the market, and $\mathbf{V}^-(t,\mathbf{x})$ is the (random) amounts of future (including present) claims payments less premiums (see e.g., Norberg [1990]). Clearly

$$\mathbf{V}^-(t,\mathbf{x}) - \mathbf{V}^+(t,\mathbf{x}) = \mathbf{x}(T) \qquad \text{for any } t \in T. \tag{2.4}$$

We are interested in the market value at any time t of $\mathbf{V}^-(t,\mathbf{x})$, given the common information $F_t$ available to the insurers at that time. This quantity we denote by $\mathbf{V}_F^+(t,\mathbf{x})$, and we may call it the *prospective F*-market value at time t, partly in accordance with actuarial terminology. Since the terms "net reserves" are usually used in connection with the quantities $\mathbf{V}^+$ and $\mathbf{V}^-$, the equations (2.2)–(2.4) justify our usage of this term for $\mathbf{x}$ in the absence of discounting.

The claims process $\mathbf{z}(t) = (z_1(t), z_2(t), \cdots, z_I(t))$ in the market we assume to be a marked, discontinuous jump process, where the marks signify the different (vector valued) claim sizes at distinct random time instants of accidents in $T$. This seems like a most natural model of claims in any insurance market. In actuarial risk theory one classical univariate example of such a process is used in the Lundberg model, where z is a compound Poisson process. Formally and more generally we assume that $\mathbf{z}$ can be represented as a stochastic integral over a random measure $\nu(\omega, A; t)$, denoted for short by $\mathbf{z} \bullet \nu$, as follows (see e.g., Gihman and Skorohod [1979a])



$$\mathbf{z}(t) = \mathbf{z} \bullet \nu = \int_0^t \iint_{R^I_+} \mathbf{u}\nu(d\mathbf{u}; ds), \qquad t \in T, \tag{2.5}$$

where $R_+ = [0, \infty)$, and the multiple state integral is over the set $R^I_+ = R_+ \times \cdots \times R_+$ (I times). Here $\nu(A; t)$ is the number of jumps the process $\mathbf{z}(s)$ makes in the time interval (0, t] with values falling in the set A, $A \in B^I_+$, where $B^I_+$ equals the Borel measurable subsets in $R^I_+$. The interpretation is that at random time points $\tau_1, \tau_2 \cdots$ events happen and a corresponding sequence of claims $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \cdots$ with values in $R^I_+$ are realized. We assume that the $(P, F_t)$-predictable intensity/transition kernel $\lambda^z(\omega, t; d\mathbf{u})$ associated with the random measure $\nu$, *the dual predictable projection* of $\nu$, can be factored into a conditional joint probability transition kernel $F^z(\omega, t; d\mathbf{u})$ and a non-negative $F_t$-predictable intensity process $\lambda^z(\omega, t)$ as follows

$$\lambda^z(t; d\mathbf{u}) = \lambda^z(t)F^z(t; d\mathbf{u}) \qquad t \in T,\ \omega \in \Omega. \tag{2.6}$$

A process is called *predictable* if it is measurable with respect to the σ-field $P(F_t)$ on $\Omega \times T$ generated by the left-continuous $F_t$-adapted processes. Intuitively a process a(t) is predictable if the values of $\mathbf{a}(t)$ can be determined from information available up to, but not including time t for each $t \in T$.

The relation between $\nu$ and $\lambda^z(t; d\mathbf{u})$ is given by

$$E\left\{\int_0^T \iint_{R^I_+} \mathbf{u}\nu(d\mathbf{u}; dt)\right\} = E\left\{\int_0^T \iint_{R^I_+} \mathbf{u}\lambda^z(t)F^z(t; d\mathbf{u})dt\right\}. \tag{2.7}$$

Let $h(t,\mathbf{u})$ be any real function such that the stochastic integral

$$\int_0^t \iint_{R^I_+} h(s,\mathbf{u})\nu(d\mathbf{u}; ds)$$

is well defined for all $t \in T$. Suppose the function is measurable with respect to $\tilde{P}(F_t) = P(F_t) \otimes B^I_+$. Any $\tilde{P}(F_t)$-measurable mapping h: $T \times \Omega \times R^I_+ \to R$ is usually called an $F_t$-predictable process indexed by $R^I_+$, or for short, predictable. For such a function we have the following

$$\int_0^t \iint_{R^I_+} h(s,\mathbf{u})\nu(d\mathbf{u};\, ds) = \sum_{n=1}^{\infty} h(\tau_n,\mathbf{u}^{(n)})1(\tau_n \le t) \tag{2.8}$$

where $1(B)$ is the indicator function of the set $B \subseteq \Omega$. By an *optional* process we mean a stochastic process measurable with respect to the $\sigma$-field $O(F_t)$ on $T \times \Omega$ generated by the right-continuous processes. The process $\mathbf{z}(t)$ defined in (2.5) is assumed to be an optional process, and $\nu$ is accordingly called an *optional random measure*. Similarly the random measure defined by (2.6) is called a *predictable random measure*, and stochastic integrals with respect to it are predictable processes given that the integrands are predictable. We only consider predictable integrands in this paper. (The stochastic integral of an optional process with respect to an optional random measure is optional.) The random measure $\nu$ can be centered, and the resulting random measure $\tilde{\nu}$ given by

$$\tilde{\nu}(A;\, t) = \nu(A;\, t) - \lambda^z(t;\, A), \tag{2.9}$$

is a martingale measure in the following sense: Let $\mathbf{z}(t)$ be any process of the type (2.5) satisfying

$$E\left(\int_0^T \iint_{R^I_-} |\,\mathbf{u}\,|^2 \lambda^z(t;\, d\mathbf{u})dt\right) < \infty. \tag{2.10}$$

Then

$$\tilde{\mathbf{z}}(t) = \int_0^t \iint_{R^I_-} \mathbf{u}\tilde{\nu}(d\mathbf{u};\, ds) \tag{2.11}$$

is a $(P, F_t)$-square integrable vector martingale. We call $(\lambda^z(\omega,t), F^z_t(\omega,d\mathbf{u}))$ the *local characteristics* of the process $\{\mathbf{z}(t),\, t \in T\}$.

The assets process $\{\mathbf{a}(t),\, t \in T\}$ in the market we assume to be a predictable process of the form

$$\mathbf{a}(t) = \mathbf{a}(0) + \tilde{\mathbf{a}}(t), \tag{2.12}$$



where $\mathbf{a}(0)$ is the initial reserve vector held at the beginning, and where $\tilde{\mathbf{a}}(t)$ equals the vector of accumulated premiums by time t, represented as a vector stochastic integral with respect to a predictable random measure $\lambda^a(t;\, d\mathbf{u})dt$. Here we use the short-hand notation $\tilde{\mathbf{a}}(t) = \mathbf{a}\bullet\lambda^a$, and we may write

$$\tilde{\mathbf{a}}(t) = \mathbf{a} \bullet \lambda^a = \int_0^t \iint_{R^I_-} \mathbf{u}\lambda^a(s)F^a(s;\, d\mathbf{u})ds, \qquad t \in T. \tag{2.13}$$

Again we have assumed that the predictable random measure $\lambda^a$ has a decomposition similar to the decomposition of the random measure $\lambda^z$ given in (2.6).

Notice that premium processes must by their very nature be predictable processes. Premiums may be paid at discrete time points, weekly, monthly, annually or otherwise, only at predictable stopping times. Alternatively premiums may be paid continually by rates, also allowed for by (2.13), a common assumption in theoretical actuarial work.

The intended interpretation of (2.13) is the following: Instead of imagining the premiums paid at time zero by a lump sum, we may assume that the premiums are paid as a stream, either at a certain rate, or at discrete future time points (i.e., at predictable stopping times). Whether or not the premiums are actually paid this way, is of no real importance. The reason that these processes are introduced will become apparent later.

### 2.3. *Preferences*

Each of the I insurers in the syndicate is represented by a pair $(\succeq^i, x_i)$, where $\succeq^i$ is a preference relation on the positive cone $L^2(\Omega,F,P)_+$, and $x_i$ is the initial portfolio of insurer i referred to above. In order to simplify the notation below, we set $X = L^2(\Omega,F,P)$, and let $X_+$ be the corresponding cone of X. A preference relation $\succeq^i$ is *x-proper* on $X_+$ if there exists some scalar $\varepsilon > 0$ and a portfolio $x \in X_+$ such that for all $y \in X_+$ the relation $y - \alpha x + z \succeq^i y$ for $z \in X$ and $\alpha \in R_+$, implies that $\|\, z \,\| \ge \alpha\varepsilon$. Here $\|\cdot\|$ is the usual $L^2$-norm $\|\, z\,\|^2 = E\{z^2(T)\}$, and our topology is the one defined by this norm. That is, the portfolio x is so desirable that z can only compensate for some loss of x if z is sufficiently large in norm (see e.g., Mas-Colell [1986]). The portfolio x in this definition is said to be *extremely desirable* for $\succeq^i$. This property of a preference relation is sometimes called *uniform properness*. We now make the following assumptions for each $i \in I$:

**Assumption 2.1:** *(i) $z \in L^2(\Omega, F, P)$ and $w > 0$ imply $z + w \succ^i z$. (ii) The graph of $\succ^i$ is relatively open. (iii) $x_M(T) = \sum_{i \in I} x_i(T)$ is extremely desirable for $\succeq^i$.*

*(iv) For all* $z \in L^2(\Omega, F, P)_+$, $\{y \in L^2(\Omega, F, P)_+ : y \succcurlyeq^i z\}$ *is convex. (v)* $x_i \in L^2(\Omega, F, P)_+$ *is not zero P a.s.*

These assumptions may be interpreted as: (i) strictly monotonic preferences, (ii) continuous preferences, (iii) the aggregate initial portfolio in the market, the market portfolio $x_M$, is extremely desirable, (iv) convex preferences, and (v) the initial portfolio of each insurer is not identically equal to zero (with probability one). From (ii) and (iii) it follows that $\succcurlyeq^i$ has a continuous utility representation $U^i(\cdot)$: $L^2(\Omega, F, P)_+ \rightarrow R$. Assumption (iii) can be considered as a smoothness condition on preferences and a strengthening of monotonicity. It holds automatically if there is a continuous, positive linear functional $\pi$ such that $\pi(y) \geq \pi(x)$ whenever $y \succcurlyeq^i x$. Conversely, if $\succcurlyeq^i$ is convex, then uniform properness implies the existence of such a functional. Thus, under risk aversion uniform properness is equivalent to a linear premium functional, which is precisely what we want.

Let us assume that there exists a market for the insurance contracts. The reinsurance syndicate Lloyd's of London used to be known for precisely this; any risk could be insured, and the market would eventually, through a negotiation process, arrive at a market premium. In order to prevent arbitrage possibilities the premium $\pi(\cdot)$ must be a linear functional on $L^2(\Omega, F, P)$. As an illustration of this point, assume on the contrary that $\pi(y_1 + y_2) > \pi(y_1) + \pi(y_2)$ for two insurance risks $y_1$ and $y_2$. Then one agent could insure the bundle $(y_1 + y_2)$ and reinsure separately $y_1$ and $y_2$. The cash flow at time zero equals $[\pi(y_1 + y_2) - \pi(y_1) - y_2)] > 0$, whereas the cash flow at the terminal date T equals $-(y_1(T) + y_2(T)) + y_1(T) + y_2(T) = 0$. This strategy leaves no obligations at the final time, so this strategy identifies a riskless profit at time zero. This is a money pump, or a "free lunch," which is inconsistent with an economic equilibrium. In the case where the inequality is reversed, the strategy is of course to insure separately $y_1$ and $y_2$ and to reinsure the bundle $(y_1 + y_2)$.

Assumption 2.1 is not necessarily the weakest that can be found. If the preference relations are represented by utility functions of the form $E\{u_i(y(T)\}$, then sufficient conditions for assumptions (i)–(iv) are that $u_i(\cdot)$ be concave, strictly increasing with a right derivative at zero and that $x_M(T)$ is bounded away from zero with probability one (Duffie [1986]). Uniqueness of equilibria in a financial economics setting is discussed by Karatzas, Lakner, Lehoczky and Shreve [1988]. In the same type of models Araujo and Monteiro [1989] have pointed out the restrictiveness of assuming that $x_M$ (or aggregate endowments) is bounded away from zero.

### 2.4. *The existence of a static equilibrium*

Here we demonstrate the existence of a competitive equilibrium with a Pareto optimal allocation. We do this by demonstrating the existence of an equilibrium in the usual Arrow-Debreu-Borch sense for the insurance economy, which we



denote by $IE = (X_+, x_i, \succcurlyeq^i; i \in I)$ where $X_+ = L^2(\Omega, F, P)_+$. For such an economy every time-state "Arrow security" is assumed available for reinsurance treaties at time zero, leaving no incentive for markets to remain open after time zero. The introduction of this static economy in a dynamic setting may seem, at first sight, to be purely a matter of ones imagination, since the number of states is uncountable. Also the dynamic feature of the economy is not transparent and can not be exploited in this framework. Nevertheless, it turns out to be a useful construction in the development of a dynamic description of an equilibrium. A key point is that we may "implement" the dynamic model in a static setting, and since we know that there exists an equilibrium in the static model, we can exploit this fact to construct an equilibrium also in the dynamic economy *without the use of dynamic programming techniques*. This last point is noticeable, since dynamic optimization for stochastic processes of the kind we are considering in our model, is a rather delicate matter which requires certain heroic assumptions on behalf of the insurers.

A static equilibrium for $IE$ is defined as a nonzero premium functional $\pi$ on X, initial portfolios $x_i \in X_+$ and reinsurance treaties $y_i \in X$ satisfying for all $i \in I$

$$\pi(y_i) \leq \pi(x_i), \tag{2.14}$$
$$v >^i y^i \Rightarrow \pi(v) > \pi(y^i) \tag{2.15}$$

and

$$\sum_{i \in I} y_i = \sum_{i \in I} x_i = x_M. \tag{2.16}$$

In a reinsurance setting this definition of an equilibrium was first formulated by Borch [1962] in a one-period model. Condition (2.14) corresponds to the budget constraint in conventional microeconomic analysis. The insurer no. i may improve his position from a risk-sharing perspective in accordance with his preferences, but the market value of his portfolio will not change (increase). Condition (2.15) states that each insurer's final portfolio is optimal according to his preferences. Condition (2.16) follows since the I insurers are assumed to exchange parts of the risks only among themselves. The following result can now be shown:

**Proposition 2.1:** *Under Assumption 2.1, IE has a static equilibrium with a Pareto optimal allocation, where the market premium functional* $\pi$ *is given by*

$$\pi(y) = E\{V(T)y(T)\}, \tag{2.17}$$

*and where* $V(T) \in X_+$. □

Since X is its own dual space, the representation in (2.17) follows in the first place from Riesz' representation theorem for linear functionals on $L^2(\Omega, F, P)$. The additional fact that $V(T) \in X_+$ requires of course separate arguments.

The proof of this result can be found in Aase [1990]. It follows by techniques which by now are becoming standard, references to which can be found in, e.g., Duffie [1986].

## 2.5. *Expected utility*

Since our goal in this paper is not to present the most general behavioral assumptions under which interesting results can be proven, but rather to present distributional realism on behalf of the risk processes in the reinsurance market and make a synthesis of this with the economic theory of uncertainty, let us specialize to the case where preferences $\succeq^i$ on $X_-$ are represented by expected utility $E\{u_i(y(T))\}$, where $u_i'(\cdot) > 0$ and $u_i''(\cdot) < 0$. Condition (2.15) can now be formulated as follows: Agent i's problem is to find a function $y(\cdot)$ such that

$$\max_{y_i(x) \in X} E\{u_i(y_i(\mathbf{x}(T))\} \tag{2.18}$$

under the budget constraint

$$\pi(y_i) = \pi(x_i), \qquad i \in I. \tag{2.19}$$

Since the premium is a linear functional on X, by the Riesz' representation theorem there exists some function $V \in X$ such that

$$\pi(x) = E\{x(T)V\} \text{ for all } x \in X. \tag{2.20}$$

We now adopt conditions guaranteeing the existence of a competitive equilibrium with an interior optimum. In the literature one such condition is known as the Inada condition (see e.g., Duffie [1988]). Denoting by $U^i(y) = E\{u_i(y(T)\}$, the technical term is that $U^i$ is additively separable and regular $(u_i)$. This involves smoothness conditions on $u_i$, and in addition it is required infinite marginal utilities at zero. Unfortunately this unboundedness is also inconsistent with the properness condition in Proposition 2.1. Given that there exists an interior solution to (2.18–19), it can be characterized as follows: Forming the Lagrangian of this problem, we seek the saddle point of

$$L(y_i; \lambda_i) = E\{u_i(y_i(\mathbf{x})) - \lambda_i(y_i(\mathbf{x}) - x_i)V\}. \tag{2.21}$$

Because of the concavity of the Bernoulli utility functions $u_i$, the necessary and sufficient conditions for an interior optimum are given by the Euler equations

$$u_i'(y_i(\mathbf{x}(T))) = \lambda_i V, \qquad i = 1, 2, \cdots, I, \qquad \text{P-a.s.} \tag{2.22}$$



The economic interpretation of $V = V(x_1(T), x_2(T), \cdots, x_I(T))$ is that it represents the market marginal utility at $\mathbf{x}(T)$. Some immediate consequences of (2.22) are:

$$V = V(x_1(T), x_2(T), \cdots, x_I(T)) = V(x_M(T)) \qquad \text{P-a.s.} \tag{2.23}$$

where $x_M(T) = \sum_{i \in I} x_i(T)$, so that only changes in the aggregate market portfolio $x_M$ affects the market marginal utility. This follows from differentiating (2.22) along $x_r$. Similarly

$$y_i(\mathbf{x}(T)) = y_i(x_M(T)), \qquad i = 1, 2, \cdots, I, \qquad \text{P-a.s.}. \tag{2.24}$$

Thus only changes in the aggregate market portfolio $x_M(T)$ affects the optimal final sharing rules $y_i$. In the static model this means that the reinsurance syndicate can hand in all their initial portfolios to a pool, and let the pool's clerk distribute parts of $x_M(0)$ back to the syndicate members according to the optimal sharing rules $y_i(x_M(0))$. Then the market closes, and reopens at time T, where $y_i(x_M(T))$ is realized by insurer i, $i = 1, 2, \cdots, I$.

Pareto optimality is established along the following lines: It is known that Pareto optimal sharing rules are found by finding functions $y_i(\cdot)$, such that the random variables $y_i(\mathbf{x}(T))$ are square integrable and solve the following

$$\max_{y_i(x) \in L^2} E\{\sum_{i \in I} k_i u_i(y_i(\mathbf{x}(T)))\} \tag{2.25}$$

such that (2.16) holds P-a.s., where $k_1, k_2 \cdots, k_I$ are arbitrary positive constants. The associated Lagrangian of this problem is

$$L(\mathbf{y}; \lambda(\mathbf{x})) = E\{\sum_{i \in I} k_i u_i(y_i(\mathbf{x}(T))) - \lambda(\mathbf{x}(T)) \sum_{i \in I} (y_i(\mathbf{x}(T)) - x_i(T))\}, \tag{2.26}$$

where the Lagrangian multiplier $\lambda(\cdot)$ is now a Borel-measurable function, meaning that $\lambda(\mathbf{x}(T))$ is an $F_T$-measurable random variable. The first order necessary and sufficient conditions for an interior optimum, given that it exists, are again given by the Euler equations

$$k_i u_i'(y_i(\mathbf{x}(T))) = \lambda(\mathbf{x}(T)), \qquad i \in I, \text{ P-a.s.}, \tag{2.27}$$

which is seen to be equivalent to (2.22) after identifying $V(\mathbf{x})$ with $\lambda(\mathbf{x})$ and $k_i$ with $\lambda_i^{-1}$. This explains why $V(\mathbf{x}(T))$ can be thought of as the shadow price per unit of P-probability when $\mathbf{x}(T,\omega) = \mathbf{x}(T)$. Thus the optimal solutions $y_i$ also satisfy collective rationality, or Pareto optimality. From the very formulation of the problem in (2.18–19) it also follows that they must satisfy individual rationality, given that this problem has a solution. Under very mild technical conditions, it was shown (see Du Mouchel [1968]) that there will always exist at least one Pareto optimal treaty. We now have the following:

**Theorem 2.1:** *Suppose the preferences* $\succcurlyeq^i$ *on* $L^2(\Omega, F, P)$ *are represented by* $U^i(y_i) = E\{u_i(y_i(T))\}$, *where* $U^i$ *is additively separable and regular* $(u_i)$. *If the aggregate market portfolio* $x_M(T)$ *is bounded away from zero with probability one, then the static reinsurance economy* IE $= (X_+, x_i, \succcurlyeq^i; i \in I)$ *has an equilibrium with a set of Pareto optimal allocations* $y_i(x_M(T))$, $i \in I$, *satisfying individual rationality. The equilibrium is characterized by (2.22) and (2.27). The premium functional is given by* $\pi(y) = E[V(x_M(T))y(x_M(T))]$, *where the market marginal utility* $V$ *is determined from the individual preferences by (2.22) and (2.23).* □

So far we have not utilized the increasing information flow $F_t$, nor have we discussed the construction of dynamic strategies producing the optimal portfolios $y_i$. In order for the theory to be useful, there must exist some strategic reinsurance treaty available to each insurer, such that the agents can adjust their net reserves in accordance with preferences as uncertainty resolves itself with time. In the next section we show how our "static," infinite dimensional problem can be reduced to a certain finite dimensional one, with an explicit dynamic description. In Section 5 we shall demonstrate the existence of optimal strategies by implementing the static model in our dynamic setting.

## 3. Dynamic equilibrium in the reinsurance economy *IE*

### *3.1. Introduction*

The economy analyzed in the preceding section is essentially a static, one-period infinite dimensional space-time decision problem. In this section we first reduce it to a dynamic $(I + 1)$-dimensional decision problem for each $t \in T$. To this end consider a net reserve process $x_j \in X$, and let $\mathbf{h}(t, \mathbf{u}) = (h_1, h_2, \cdots, h_I)(t, \mathbf{u})$ be a predictable, $R^I_-$-indexed process satisfying

$$E\left[\int_0^T \iint_{R^I_+} |\, \mathbf{x}\cdot\mathbf{h}(t, \mathbf{u})\,|^2 \lambda^z(t) F_t^z(d\mathbf{u})dt\right] < \infty. \qquad (3.1)$$

We define the set $L^2(\mathbf{x})$ to be the following:

$$L^2(\mathbf{x}) = \{\mathbf{h};\ \mathbf{h}(t, \mathbf{u}) \text{ is predictable and } R^I_-\text{-indexed, satisfying (3.1)}\} \qquad (3.2)$$



For any process $\mathbf{h} \in L^2(\mathbf{x})$ the stochastic integral $\int_{[0,t]} \iint_{R^I_-} \mathbf{h}(s,\mathbf{u})\bar{\nu}(d\mathbf{u}; ds)$ is well defined, and is a $(P, F_t)$-square integrable martingale.

*3.1.1.* ***Proportional treaties.*** Now, with some abuse of notation, let $\mathbf{h}(t,\omega) = \mathbf{h}(t,\mathbf{x})$ be a $\tilde{P}(F_t)$-measurable process. For the model in Section 2 the following stochastic integral is also well defined

$$\int_{[0,t]} \mathbf{h}(s,\mathbf{x})d\mathbf{x}(s) = \int_{[0,t]} \mathbf{h}(s,\mathbf{x}(s)) \cdot \iint_{R^I_+} \mathbf{a}\lambda_s^a F_s^a(d\mathbf{a})ds$$
$$- \int_{[0,t]} \mathbf{h}(s,\mathbf{x}(s)) \cdot \iint_{R^I_-} \mathbf{z}\nu(d\mathbf{z}; ds). \qquad (3.3)$$

Here we interpret $h_i(t, \mathbf{x})$ as the fraction of the initial portfolio $x_i(t)$ held by some insurer at time t, if the net reserves in the market at this time equal $\mathbf{x}(t)$. This strategy satisfies the budget constraints of the insurer if

$$\mathbf{h}(t, \mathbf{x})\mathbf{x}(t) = \mathbf{h}(0, \mathbf{x})\mathbf{x}(0) + \int_0^t \mathbf{h}(s, \mathbf{x})d\mathbf{x}(s) \qquad \text{for all } t \in T. \qquad (3.4)$$

This requirement here means that the final value equals the initial value plus any gains and losses incurred from proportional reinsurance treaties settled in (0, t), new risks undertaken and claims incurred following the strategy $\mathbf{h}$. Note that a single contract need not be "self-financing," since settled claims may require payouts, and premiums may be paid in on a weekly, monthly or annual basis, as a rate or otherwise. The equation (3.4) is a budget constraint on the total activity of an agent.

It follows from the product formula that (3.4) implies that

$$\int_0^t \mathbf{x}(s-)d\mathbf{h}(s, \mathbf{x}) = 0 \qquad \text{for all } t \in T. \qquad (3.5)$$

Returning to equation (3.3), the stochastic integral there may be interpreted as the gains and losses from following a proportional reinsurance strategy $\mathbf{h}$ in [0, t]. Let $\mathbf{h}^{(i)}(t, \mathbf{x})$ be insurer i's strategy, $i = 1, 2, \cdots, I$. By definition $h_j^{(i)}(0, \mathbf{x}) = \delta_{ij}$

(Kronecker's delta) and any admissible (i.e., square integrable) reinsurance strategy $\mathbf{h}^{(i)}$ must satisfy

$$\sum_{j\in I} h_j^{(i)}(t, \mathbf{x})x_j(t) = \int_{[0,t]} \sum_{j\in I} h_j^{(i)}(s, \mathbf{x})dx_j(s)$$

$$= x_i(0) + \int_0^t \sum_{j\in I} h_j^{(i)}(s, \mathbf{x})dx_j(s),\ i\in I,\ t\in T. \tag{3.6}$$

Let $M$ be the set of reinsurance contracts that can be generated this way. In general $M$ is a closed subspace of $L^2(\Omega, F, P)$, which follows from the Kunita-Watanabe inequality: Convex combinations of proportional reinsurance treaties are again proportional treaties.

### 3.2. *Market premiums*

In this economy we first take as a numeraire a pure discount bond which pays at the terminal date T one unit of account. This asset we denote by $x_0$, where $x_0(t) \equiv 1$ for all $t\in T$. As a consequence the market premium of $x_0$, $\pi(x_0) = 1$, or, stated differently, $\pi(x_0) = E\{V(T)x_0(T)\} = E\{V(T)\} = 1$, where V(T) is the same as in (2.17). Also $V(T,\omega) \geq 0$ P-a.s., which means that we can define a new, bona fide probability measure P* by

$$P^*(A) = \int_A V(T,\omega)dP(\omega), \qquad \frac{dP^*}{dP} = V(T), \qquad A\in F. \tag{3.7}$$

Here V(T) equals the Radon-Nikodym derivative of P* with respect to P. Denoting the expectation operator under P* by E*, it follows from (3.7) that the market premium at time zero of any portfolio $x \in X$ can be written

$$\pi(x) = E\{V(T)x(T)\} = E^*\{x(T)\}. \tag{3.8}$$

The prospective $F$-market value at time t, $\mathbf{V}_F^+(t,\mathbf{x}) = E^*\{\int_{[t,T]} d(-\mathbf{x}(s)) \mid F_t\} = -E^*\{\mathbf{x}(T) - \mathbf{x}(t-) \mid F_t\}$, and from assumption (1.2) it follows by Hölder's inequality that $E^*\{\mathbf{x}(T) \mid F_t\}$ is a $(P^*, F_t)$-martingale. The market value of the accumulated future claims payments less premiums can alternatively be expressed using the given probability measure P and the market marginal utility V(T). Define the market marginal utility process by



$$V(t) = E\{V(T) \mid F_t\},\ t \in T,\ \text{P-a.s.} \tag{3.9}$$

It follows that V(t) can be interpreted as the "spot price of risk" at each time instant t. Using the notation $\pi(\mathbf{x})(t) = -\mathbf{V}_F^+(t,\mathbf{x})$, it now follows that $\pi(\mathbf{x})(t) = E^*\{\mathbf{x}(T) - \mathbf{x}(t-) \mid F_t\} = E^*\{\int_{[t,T]} d(\mathbf{x}(s)) \mid F_t\} = E\{\frac{1}{V(t)}\int_{[t,T]} V(s)d\mathbf{x}(s) \mid F_t\} = E\{\frac{V(T)}{V(t)}(\mathbf{x}(T) - \mathbf{x}(t-)) \mid F_t\} = \frac{1}{V(t)}E\{V(T)\mathbf{x}(T) \mid F_t\} - \mathbf{x}(t-)$. The third equality follows from the change of measure, the last from the definition (3.9) and the adaptedness of the processes $\mathbf{x}(t)$ and V(t). The fourth equality follows from a result due to Dellacherie and Meyer [1976]: Let $A_t$ be an increasing process adapted to $F_t$, and let $M_t$ be a nonnegative $(P, F_t)$-martingale, right-continuous and uniformly integrable. Then for any $F_t$-stopping time $\tau$ the following holds true

$$E\left\{\int_0^\tau M(t)dA(t)\right\} = E\left\{M(\tau)A(\tau)\right\}. \tag{3.10}$$

Thus we can write

$$E^*\{\mathbf{x}(T) \mid F_t\} = \frac{1}{V(t)}E\{V(T)\mathbf{x}(T) \mid F_t\},\ t \in T,\ \text{P-a.s.} \tag{3.11}$$

It follows in the same way that the market premium of any risk y at time t, i.e. of the remaining risk in [t, T] equals

$$\pi(y)(t) = \frac{1}{V(t)}E\{\int_{[t,T]} V(s)dy(s) \mid F_t\} = E^*\{y(T) \mid F_t\} - y(t-). \tag{3.12}$$

### 3.3. *The representation property for martingales*

The theoretical results of the preceding sections allow us to conclude that the market value of an insurer's net reserves must depend upon (i) the stochastic properties of his net reserves, (ii) the stochastic relationship between his particular reserves and the accumulated reserves in the market and (iii) the attitude towards risk in the market. Starting with a set of utility functions for the individual insurers, it seems rather complicated to derive explicitly a formula for $V(x_M(T))$

which take account of all the relevant inputs to this stochastic reinsurance economy $IE_S$. In this subsection this task will nevertheless be our primary first goal.

The dynamics of the process $\mathbf{x}(t)$ is given by (2.5) and (2.12–13), where

$$\mathbf{x}(t) = \mathbf{a}(t) - \mathbf{z}(t), \tag{3.13}$$

$\mathbf{a}(t)$ being a predictable random measure, $\mathbf{z}(t)$ being an optional random measure. Let P* be the probability measure, absolutely continuous with respect to P, under which $\nu$ has the dual predictable projection $\lambda^a(t; \mathbf{du})dt$, or phrased differently, under which the claims process $\mathbf{z}(t)$ has local characteristics $(\lambda^a(t), F^a(t; \mathbf{du}))$. In this case we get by Girsanov's theorem for random measures (see e.g., Jacod and Shiryaev [1987, (5.5), p. 179]) that

$$\mathbf{x}(t) \overset{P^*}{=} \mathbf{x}(0) + \mathbf{a} \bullet \lambda^a - \int_0^t \iint_{R^I_+} \mathbf{u}\, \lambda^a(\omega; s)F^a(\omega; s, \mathbf{du})ds - \mathbf{z} \bullet \tilde{\nu} \tag{3.14}$$

where as usual $\tilde{\nu}$ is the *centered* random measure under P*. From equation (2.13) it follows that the second and the third term on the right-hand side of equation (3.14) cancel. (The third term is analogous to the additional drift-term in Girsanov's theorem for diffusion processes.) The conclusion is that $\mathbf{x}(t)$ is a $(P^*, F_t)$-martingale, since the last term is a martingale by Hölder's inequality (recall equations (2.10–11)). We now make the following

**Assumption 3.1:** *There exists a unique pair of processes $\mu$ $(\omega; t)$ and $\nu(\omega; \mathbf{u}, t)$ satisfying the following:*

*(I) $\mu$ $(\omega; t)$ is a predictable, non-negative process satisfying*

$$\lambda^a(\omega; t) = \mu(\omega; t)\lambda^z(\omega; t), \qquad t \in T, \text{ P-a.s.}, \tag{3.15}$$

*such that*

$$\int_0^T \mu(\omega; t)\lambda^z(\omega; t)dt < \infty, \qquad \text{P-a.s.} \tag{3.16}$$

*(II) $\nu(\omega; \mathbf{u}, t)$ is a predictable process satisfying*

$$F^a(\omega; t, \mathbf{du}) = \nu(\omega; \mathbf{u}, t)F^z(\omega; t, \mathbf{du}), \qquad t \in T, \mathbf{u} \in R^I_+, \text{ P-a.s.}, \tag{3.17}$$



*such that*

$$\iint_{R^I_+} \nu(\mathbf{u}, t)F^z(t, \mathbf{du}) = 1 \text{ for all } t \in T, \text{ P-a.s.} \tag{3.18}$$

Under Assumption 3.1 there exists a probability measure P* absolutely continuous with respect to the original probability measure P, such that the above martingale property holds, and such that the local characteristics of the random, marked vector process $\mathbf{z}$ under P* is given by $(\lambda^a(\omega; t), F^a(\omega; \mathbf{du}))$. Assumption 3.1 is not some set of harmless technical requirements. It yields unique premiums on $M$. In other words, the vector of income processes satisfying this assumption results in unique pricing in an otherwise incomplete model: Consider the informational constraint

$$F_t = F_0 \vee F^\nu_t, \tag{3.19}$$

where $F^\nu_t$ is the internal history of the process $\{\mathbf{z}(t), t \in T\}$ with associated random measure $\nu$. Under P*

$$\mathbf{x}(t) \overset{P^*}{=} \mathbf{x}(0) - \int_0^t \iint_{R^I} \mathbf{u}\tilde{\nu}(\mathbf{du}; ds), \tag{3.20}$$

which means that (3.19) and (1.1) coincide. It follows from results in Boel, Varaiya and Wong [1975] and Jacod [1975] that any $(P^*, F_t)$-martingale $y(t)$ can be represented as follows:

$$y(t) = y(0) + \int_0^t \iint_{R^I_+} \sum_{i \in I} u_i h_i(s, \mathbf{u})\tilde{\nu}(\mathbf{du}; ds), \qquad t \in T, \tag{3.21}$$

for some uniquely determined predictable process $\mathbf{h}(t) \in L^2(\mathbf{x})$. In order to formulate the conditions under which dynamic spanning of the proportional kind may hold in our model, we here recall a somewhat more detailed specification of a predictable function: Consider the set of functions $\mathbf{h}(t,\mathbf{u}) = \mathbf{h}(t,\omega,\mathbf{u})$ of the form $\mathbf{h}(t,\omega,\mathbf{u}) = \sum_{i=0}^{k} \mathbf{h}_i(t,\omega)\phi_i(\mathbf{u})$ where the $\phi_i$ are bounded functions measurable with respect to $B^I_+$ and where $\mathbf{h}_i$ are bounded, predictable processes. Then a function $\mathbf{h}(t,\mathbf{u}) = \mathbf{h}(t,\omega,\mathbf{u})$ is said to be predictable if there exists a sequence $\mathbf{h}^{(i)}$ of functions of the above kind such that $\lim_{j\to\infty} \mathbf{h}^{(j)}(t,\omega,\mathbf{u}) = \mathbf{h}(t,\omega,\mathbf{u})$ for all $(t,\omega,\mathbf{u}) \in T \times \Omega \times R^I_+$. Suppose for any martingale y the function $\mathbf{h}(t,\omega,\mathbf{u})$ is determined by equation (3.21) and suppose that the functions $\phi_i^{(j)}$ in the determination of $\mathbf{h}(t,\omega,\mathbf{u})$ satisfies

$$\int_{(0,t]} \iint_{R^I_+} \mathbf{h}(s,\mathbf{u}) \cdot \mathbf{u}\tilde{\nu}(d\mathbf{u};\, ds)$$

$$= \int_{(0,t]} \lim_{j\to\infty} \sum_{i=1}^{k_j} \mathbf{h}_i^{(j)}(s,\omega) \cdot \iint_{R^I_+} \phi_i^{(j)}(\mathbf{u})\mathbf{u}\tilde{\nu}(d\mathbf{u};\, ds) \tag{3.22}$$

$$= \int_{(0,t]} \mathbf{h}(s,\omega) \cdot \iint_{R^I_+} \mathbf{u}\tilde{\nu}(d\mathbf{u};\, ds).$$

In this case our model can be described by proportional treaties only, and

$$y(t) = y(0) + \int_0^t \mathbf{h}(s)d\mathbf{x}(s),\ t \in T. \tag{3.23}$$

The property (3.21) is often called the representation property of y relative to $\mathbf{x}$ under $P^*$ in probability theory, but for our purposes we see that this terminology may be somewhat misleading, since we should like to reserve this property for the cases where the equation (3.23) holds instead. The terms $h_i(t,\mathbf{u})$ in (3.21) involve non-linear reinsurance on the sizes $\mathbf{u}^{(m)}$ of claims in the market. If (3.23) holds true, *any* reinsurance treaty can be obtained from proportional ones. Under Assumption 3.1, if (3.21) and (3.23) coincide, then $M = H$, where $H$ is the space of square integrable $P^*$-martingales. In general is the representation in (3.23) more restrictive than the representation result (3.21), so that $M \subset H$. Restricting attention to the set of proportional contracts $M$, this does not necessarily only limit the reinsurers to proportional treaties, as may be indicated by (3.23). The point is that *any* reinsurance treaty in $M$, nonlinear or otherwise, can, at least in principle, be attained by a suitable proportional dynamic risk sharing strategy with continuous trading.

*3.3.1. **Non-proportional reinsurance treaties***. Suppose now that $M \neq H$. Non-proportional treaties such as excess of loss or stop loss reinsurance do exist in the market and can be traded directly. The representation (3.21) involves such treaties. Since transaction costs can be substantial in reinsurance markets due to the extensive usage of professional brokers, clearly a proportional representation such as (3.23), if true, is mainly of theoretical interest. However, there may exist treaties, wanted by some reinsurer, that is not available for direct trade. By the representation (3.21) the reinsurer could in principle manufacture this treaty himself by a suitable dynamic risk sharing strategy. Now, since $M$ is in general a closed subspace of the Hilbert space $H$, we know that there exists a unique pair of linear mappings f and g such that any $y \in H$ can be written as $y = f(y) + g(y)$, f maps $H$ into $M$, g maps $H$ into $M^\perp$, where $M^\perp$ is the set of all $y \in H$ which are orthogonal to every $z \in M$. By orthogonal is here meant that $E^*(z_T y_T) = 0$. If $y \in M$, then $f(y) = y$, $g(y) = 0$; if $y \in M^\perp$, then $f(y) = 0$, $g(y) = y$. Also (i) $\| y - f(y) \| = \inf\{\| y - z \| : z \in M\}$ if $y \in H$. (ii) $\| y \|^2 = \| f(y) \|^2 + \| g(y) \|^2$ (here again norm is with respect to $E^*$). Now, if y is some contract that can not be represented as in (3.23), but if $\mathbf{x}$ is a vector martingale with respect to $P^*$, a projection theorem due to Kunita and Watanabe [1967] allows us to replace (3.23) by



$$y(t) = y(0) + \int_0^t \mathbf{h}^S(s) \cdot d\mathbf{x}(s) + m(t) \qquad P^*\text{-a.s.}, \tag{3.24}$$

where m(t) is a martingale which is $P^*$ orthogonal to $\mathbf{x}$, $m(0) = 0$. Corresponding to the general theory, $m(t) \in M^\perp$, or $g(y) = m$ and $f(y) = \int \mathbf{h}^S \cdot d\mathbf{x}$. Following Föllmer and Sondermann [1986], we can define a "cost process" by $C_t(\mathbf{h})^S = \mathbf{h}^S(t) \cdot \mathbf{x}(t) - \int_0^t \mathbf{h}^S(s) \cdot d\mathbf{x}(s)$, and it is clear that under proportional spanning $C_t(\mathbf{h}^S) = \mathbf{h}^S(0) \cdot \mathbf{x}(0)$, which is the "arbitrage price" of y. Föllmer and Sondermann show that if the strategy $\mathbf{h}^S$ is y-admissible and the cost process $C_t(\mathbf{h}^S)$ is a $(P,F_t)$-martingale, then there exists a unique proportional strategy $\mathbf{h}^{S,y}$ which at each time $t \in [0,T]$ minimizes the following measure $R_t(\mathbf{h})$ of expected remaining cost $E^*\{(C_T(\mathbf{h}) - C_t(\mathbf{h}))^2 \mid F_t\}$. In general terms we could say that it is possible for any general, non-proportional treaty $(y, \mathbf{h}(t,\mathbf{u}))$ to find a unique proportional treaty $\mathbf{h}^S(t)$ corresponding to an element in $M$ as "close as possible" to y, where as close as possible also could involve minimizing the measure of risk $R_t(\mathbf{h})$. The results of Föllmer and Sondermann have been generalized to the case where $\mathbf{x}$ is a semimartingale under $P^*$ by Schweitzer [1988].

Under Assumption 3.1 there exists a unique V, to be determined in the next section, such that our pricing results are still valid. This follows since $\pi(y) = E^*(y(T)) = y(0)$, as $E^*(m(T)) = 0$ since m is a $P^*$-martingale which is zero at time zero. We may consider the term m as a "nonsystematic" part of the overall risk y. This term can obviously be written

$$m_t = \int_0^t \left( \iint_{R^I_+} \mathbf{h}(s,\mathbf{u}) \cdot \mathbf{u}\tilde{\nu}(d\mathbf{u};\, ds) - \mathbf{h}^S(s)d\mathbf{x}(s) \right). \tag{3.25}$$

i.e., as a difference between a non-proportional and a proportional treaty, both $(P^*, F_t)$-martingales.

### 3.4. *The market marginal dis-utility process*

Returning to Section 3.2, we can now give a characterization of the Radon-Nikodym derivative $V(T) = \frac{dP^*}{dP}$ in (3.9) for the measure P* in Assumption 3.1. Appealing again to results in Boel, Varaiya and Wong [1975] and Jacod [1975], we have the following *density process* for our model

$$V(t) = \left(\prod_{n\geq 1} \mu(\tau_n)v(\mathbf{z}^{(n)},\tau_n)I(\tau_n \leq t)\right) \cdot \exp\left\{\int_0^t \iint_{R^I_+} [1 - \mu(s)v(\mathbf{u},s)]\lambda_s^z F_s^z(d\mathbf{u})ds\right\}, \ t\in T. \tag{3.26}$$

(the product $\Pi_{n \geq 1}$ is taken to be 1 if $\tau_1 > t$). Since $E\{V(T)\} = 1$, it can be shown using the Doléans-Dade exponential formula that $V(t) = E\{V(T) \mid F_t\}$ P-a.s., so that V(t) is indeed a $(P, F_t)$-martingale over $T$.

We now turn to the economic interpretation of the two new terms appearing in (3.26) as well as in Assumption 3.1: The process v is a random measure under P* as well, and the claims process $\{\mathbf{z}(t), t\in T\}$ is still a random, marked point process under P*, but now with local characteristics $\{\mu_t\lambda_t, v(\mathbf{z}, t)F^z(t, d\mathbf{z})\}$. Combining this with Theorem 2.1, we have that the function $v(\mathbf{z}, t) = v(z_M, t)$ for all $t \in T$, where $z_M = \Sigma_{i\in I}z_i(t)$. Furthermore, since the function $v(\cdot, t)$ is itself a Radon-Nikodym derivative of the new conditional claim size distribution under P* with respect to the old one under P, $v(z_M, t)$ can be interpreted as the *marginal dis-utility of accumulated claim sizes in the market at time* t. In a market of only risk neutral members, $v(z_M, t) \equiv 1$ for all $(\mathbf{z}, t) \in R^I_+ \times T$.

Jumps, or claim sizes are not the only source of uncertainty in this model. The time instants $\tau_n(\omega)$ when accidents occur are random as well and the frequency is measured by $\lambda^z(\omega, t)$, which is itself allowed to be an $F_t$-predictable stochastic process. The process $\mu(\omega, t)$ thus measures *the attitude towards frequency risk in the market* at each time $t \in T$. In actuarial terminology one may perhaps say that $\mu$ corresponds to a loading on frequency. We return to specific examples in Section 7.

Notice that the above is formulated in terms of marginal dis-utilities on claim sizes. A natural reformulation can be given in terms of marginal utilities on net reserves, or $v(z_M, t) = \bar{v}(a_M(t) - z_M, t)$. Below it will be convenient to work with



v instead of $\bar{v}$, but we should keep in mind that the market premiums also depends on $a_M(t)$ = the total initial reserves at time t allocated to the line of reinsurance under consideration.

Since the $\mathbf{\bar{a}}(t)$-process of assets in the market stem from incomes through market premiums, clearly this vector stochastic process must depend upon market preferences, and thus be determined endogenously. The precise form this takes in a stochastic equilibrium is given in (3.15) and (3.17), with the above interpretation of the terms $v(\cdot)$ and $\mu(\cdot)$. The linkage of $\mathbf{\bar{a}}(t)$ to the market now follows, since the pair $(\mu, v)$ in (3.15–18) is the same as the functions appearing in the expression for V(T) given in (3.26). Since V(T) is determined by the market participants, so is $\mathbf{\bar{a}}(t)$. This should also be compared to the classical Lundberg model of an insurance company, where the incomes process is given exogenously as a linear function $a(t) = ct$, where $c > 0$ is a constant. In this latter case no economic theory is used, and only one company is considered (Lundberg [1926]). It is remarkable, though, that this theory was developed before the theory of stochastic processes was formally established.

### 3.5. *Stochastic equilibria*

Let y be any portfolio of any of the reinsurers. We call a reinsurance strategy $\mathbf{h}(t,\mathbf{u}) \in L^2(\mathbf{x})$ *budget feasible* if

$$y(t) = \int_{[0,t]} \iint_{R^I_+} \mathbf{h}(s,\mathbf{u}) \cdot \mathbf{u}\lambda^a(s)F^a(s;d\mathbf{u})ds - \int_{[0,t]} \iint_{R^I_+} \mathbf{h}(s,\mathbf{u}) \cdot \mathbf{u}\nu(d\mathbf{u};ds), \qquad t\in T. \tag{3.27}$$

and it is *optimal* for insurer i provided there is no other budget feasible strategy $(w, \mathbf{g})$ such that $w >^i y$. Notice that the predictability requirement means that $\mathbf{h}_t$ gives the position of the reinsurer after the last treaty adjustment *before* time t, but $\mathbf{h}_t$ does not depend upon the values of $\mathbf{x}(t)$ *at* time t.

A stochastic reinsurance economy $IE_S$ is now defined by the triplet $IE_S = (IE, \{F\}, L^2(\mathbf{x}))$, and a *stochastic equilibrium* for $IE_S$ is a collection $(\pi, x_i, y_i, \mathbf{h}^{(i)}(t,\mathbf{u});\ i\in I, t\in T)$ where $\pi$ is a market premium functional for $IE_S$, $(y_i, \mathbf{h}^{(i)}(t,\mathbf{u});\ i\in I, t\in T)$ is an optimal reinsurance strategy for each insurer $i \in I$ such that markets clear P-a.s.:

$$\sum_{i\in I} y_i(t) = \sum_{i\in I} x_i(t), \qquad t \in T, \tag{3.28}$$

and

$$\sum_{i \in I} h^{(i)}(t,u) = 1, \qquad t \in T, \; u \in R^l_+. \tag{3.29}$$

The dynamic type of equilibriums, of which the above is a special case, were first studied by Radner [1972]. In Section 5 we demonstrate the existence of such an equilibrium in our reinsurance syndicate.

## 4. The individual dynamic optimization problems—proportional treaties

### *4.1. Introduction*

In this section we present the individual insurers' dynamic optimization problems in some detail. As is usually the case with dynamic programming, a set of rather strong assumptions is required in order for the problem to be guaranteed a solution. Instead of concentrating on the economic contents of such conditions in this section, we defer to Section 5 to demonstrate sufficient conditions for the existence of a stochastic equilibrium. Therefore our presentation in this section is somewhat heuristic. We shall only be considering optimal *proportional* treaties in what follows.

### *4.2. Itô's generalized lemma and the Hamilton-Jacobi-Bellman equations*

Each agent $i \in I$ has to solve the following:

$$\max_{h^{(i)} \in H(x)} E\{u_i(y_i(T))\} \tag{4.1}$$

subject to the budget constraints

$$y_i(t) = \sum_{j \in I} h_j^{(i)}(t,x)x_j(t) = x_i(0) + \int_0^t \sum_{j \in I} h_j^{(i)}(s,x)dx_j(s), \qquad t \in T. \tag{4.2}$$

where $H(x)$ is the set of permissible, square integrable proportional strategies $h^{(i)}$ such that the budget constraint (4.2) holds. Let $y_i = y_i(t)$ and define for each reinsurer $i \in I$ the following *indirect utility function*

$$Z_i(t, y_i) = \sup_{h^{(i)} \in H(x)} E\{u_i(y_i(T)) \mid F_t\}, \text{ for all } t \in T. \tag{4.3}$$



Notice that the budget constraint (3.4) together with (3.6) yield the present budget constraint (4.2). Clearly the indirect utility of each insurer must depend upon the market preferences, since the transactions implicit in (4.2) are assumed to take place at market prices.

Under certain smoothness conditions the functions $Z_i(t,y_i)$ may satisfy a nonlinear equation of parabolic type, a generalized version of the Hamilton-Jacobi-Bellman equation: Starting with the model for the net reserves in the market

$$x(t) = x(0) + \int_0^t \iint_{R^l_+} a(\mu_\zeta v(a,s) - 1)\lambda_\zeta^\zeta F_\zeta^\zeta(da)ds - \int_0^t \iint_{R^l_+} z\tilde{\nu}(dz; ds), \qquad t \in T, \tag{4.4}$$

we need Itô's generalized lemma, which in our model takes the following form: let $G(t, x)$ be some continuously differentiable function in the first argument, twice continuously differentiable in the second. Then (see e.g., Gihman and Skorohod [1979 a-b])

$$G(t,x(t)) = G(0,x(0)) + \int_0^t \frac{\partial G}{\partial s}(s,x(s))ds + \int_0^t \iint_{R^l_+} \sum_{j \in I} \frac{\partial G}{\partial x_j}(s,x)a_j(\mu_\zeta v(a,s) - 1)\lambda_\zeta^\zeta F_\zeta^\zeta(da)ds + \int_0^t \iint_{R^l_+} L^d(G,z)(s)\lambda_\zeta^\zeta F_\zeta^\zeta(dz)ds - \int_0^t \iint_{R^l_+} \Big[G(s-,x(s-) - z) - G(s-,x(s-))\Big]\tilde{\nu}(dz; ds), \tag{4.5}$$

where

$$L^d(G,z)(s) = \Big[G(s,x(s-) - z) - G(s,x(s-)) + \sum_{j \in I} \frac{\partial G}{\partial x_j}(s,x(s-))z_j(s)\Big], \quad s \in T. \tag{4.6}$$

Using this we may formally proceed as follows: Agent i's position at time t equals

$$y_i(t) = \sum_{j\in I} x_j(t) h_j^{(i)}(t,\mathbf{x})$$

$$= x_i(0) + \int_0^t \sum_{j\in I} h_j^{(i)}(s,\mathbf{x})dx_j(s)$$

$$= x_i(0) + \int_0^t \sum_{j\in I} \Big\{ \iint_{R_+^I} h_j^{(i)}(s,\mathbf{x})[a_j(\mu,\nu(\mathbf{a},s) - 1)\lambda_s^z F_s^z(d\mathbf{a})ds \qquad (4.7)$$

$$- z_j\tilde{\nu}(d\mathbf{z};\, ds)]\Big\}, \qquad t \in T,\ i \in I.$$

Utilizing the generalized Itô formula (4.5–6) and assuming that the functions $Z_i(t,y)$ possess all the properties required for the applicability of this formula, we have the following

$$Z_i(t,\, y_i(t)) - Z_i(s,\, y_i(s)) = \int_s^t LZ_i^{h^{(i)}}(u,y_i(u))du + \int_s^t \Lambda Z_i^{h^{(i)}}(u,y_i(u),du), \qquad (4.8)$$

where the first term on the right-hand side is the following:

$$LZ_i^{h^{(i)}}(t,\, y_i(t)) = L^{\mu\nu}Z_i^{h^{(i)}}(t,\, y_i) - L^{\nu}Z_i^{h^{(i)}}(t,\, y_i), \qquad (4.9a)$$

where

$$L^{\mu\nu}Z_i^{h^{(i)}}(t,\, y_i) = \iint_{R_+^I} \frac{\partial Z_i}{\partial y_i}(t,y_i) \sum_{j\in I} h_j^{(i)} a_j(\mu,\nu(\mathbf{a},t) - 1)\lambda_t^z F_t^z(d\mathbf{a}), \qquad (4.9b)$$

and

$$L^{\nu}Z_i^{h^{(i)}}(t,\, y_i) = \iint_{R_+^I} \Big[ Z_i\Big(t,\, y_i - \sum\nolimits_{j\in I} h_j^{(i)}(t,\mathbf{x})z_j\Big) - \qquad (4.9c)$$

$$Z_i(t,\, y_i) + \frac{\partial Z_i}{\partial y_i}(t,\, y_i)\sum\nolimits_{j\in I} h_j^{(i)}(t,\mathbf{x})z_j\Big]\lambda_t^z F_t^z(d\mathbf{z}).$$



The last term in (4.8) is a martingale, where

$$\Lambda Z_i^{\mathbf{h}^{(i)}}(t,\, y_i(t),dt) = \iint_{R_+^I} \Big[ Z_i\Big(t,\, y_i(t) - \sum\nolimits_{j\in I} h_j^{(i)}(t,\mathbf{x})z_j\Big)$$

$$- Z_i(t,\, y_i(t))\Big]\tilde{\nu}(d\mathbf{z};dt). \qquad (4.9d)$$

Therefore

$$\lim_{t\downarrow s} \frac{1}{t-s} E\Big\{Z_i(t,\, y_i(t)) - Z_i(s,\, y_i(s)) \mid F_s\Big\} = L\, Z_i^{\mathbf{h}^{(i)}}(s,\, y_i(s)). \qquad (4.10)$$

We now use the optimality principle, which in the present situation takes the following form:

$$Z_i(s,\, y_i) \geq E\{Z_i(t,y_i(t)) \mid F_s\},\ t \geq s, \text{ where } y_i(s) = y_i, \qquad (4.11)$$

with equality sign valid if and only if the strategy $\mathbf{h}^{(i)}$ is optimal. For $s < t$ we have in view of this

$$\frac{1}{t-s}[Z_i(s,\, y_i) - Z_i(t,\, y_i)]$$

$$= \sup_{\mathbf{h}^{(i)}\in H} E\Big\{\frac{1}{t-s}[Z_i(t,\, y_i(t)) - Z_i(t,\, y_i)] \mid F_s\Big\}. \qquad (4.12)$$

By taking limits as $t \downarrow s$, the left-hand side converges to $-\frac{\partial Z_i}{\partial t}$. By interchanging the order of passage to the limit and the operation of taking l.u.b., we use (4.10) above and arrive at the following equation,

$$-\frac{\partial Z_i}{\partial t}(t,\, y_i) = \sup_{\mathbf{h}^{(i)}\in H} L_{\mathbf{h}^{(i)}} Z_i(t,\, y_i),\ t\in[0,T), \qquad (4.13)$$

where $L_{\mathbf{h}^{(i)}} Z_i(t,\, y_i) = LZ_i^{\mathbf{h}^{(i)}}(t,\, y_i)$. The equation (4.13) has the terminal condition

$$Z_i(T,\, y_i) = \sup_{\mathbf{h}^{(i)}\in H} u_i\Big(y_i(T)\Big). \qquad (4.14)$$

### 4.3. *Optimal Markovian strategies*

Suppose there exist functions $Z_i(t, y)$, $t \in T$, $i = 1, 2, \cdots, I$, continuously differentiable with respect to t and twice continuously differentiable with respect to the second argument satisfying the boundedness conditions $|Z_i(t, y)| \le C(1 + y^2)$, $\left|\frac{\partial}{\partial y} Z_i(t, y)\right| \le C(1 + |y|)$, $\left|\frac{\partial^2}{\partial y^2} Z_i(t, y)\right| \le C$, $i \in I$, and the generalized Hamilton-Jacobi-Bellman equations (4.13) as well as the terminal condition (4.14). Then under the above stipulated conditions the functions $Z_i(t,y)$ coincides with the indirect utility functions of each of the insurers using admissible strategies $\mathbf{h}^{(i)} \in H(\mathbf{x})$. Moreover there exists a set of I optimal Markovian strategies $\mathbf{h}^{(i)}(t,\mathbf{x}) = \phi^{(i)}(t,\mathbf{x}(t))$ provided there exist Borel functions $\phi^{(i)}(t,\mathbf{x})$, $(t,\mathbf{x}) \in T \times R^I_+$ satisfying the equation

$$\sup_{\mathbf{h}^{(i)} \in H} L_{\mathbf{h}^{(i)}} Z_i(t, y_i) = L_{\phi^{(i)}(t,\mathbf{x})} Z_i(t, y_i). \tag{4.15}$$

If this holds true, the Hamilton-Jacobi-Bellman equation can be written

$$-\frac{\partial Z_i(t, y_i)}{\partial t} = L_{\phi^{(i)}(t,\mathbf{x})} Z_i(t, y_i), \qquad i = 1, 2, \cdots, I, \tag{4.16}$$

subject to the usual terminal condition (4.14). As soon as we have solutions to the system of equations (4.16), it is relatively easy to construct the optimal proportional strategies $\phi^{(i)}(t,\mathbf{x}(t))$ for each of the I insurers from the insurers' indirect utility functions $Z_i$. Just solve the maximization problem (4.15) for each $i \in I$, that is, the original dynamic optimization problem under uncertainty is reduced to the easier problem of finding the maximum of a real function defined on $R^I$.

Notice that the term $L^{\mu\nu}$ contains the market attitude towards frequenty risk $\mu$ and towards claim size risk $\nu$, so that the optimal strategies $\phi^{(i)}$ must also depend upon market preferences. This is quite natural, since the premiums that the agents face at each transaction depend upon the market attitude towards risk, as we have pointed out earlier.

In the cases where we know that a set of optimal strategies $\phi^{(i)}(t,\mathbf{x}(t))$ exist for the I agents, then we can use (4.15–16) to construct this set of strategies. In the next section we demonstrate the existence of optimal strategies.

In the Markovian case treated in this subsection we could alternatively have used the infinitesimal generator approach.

For an example of the solution of an equation of the type (4.16) subject to boundary conditions found for R&D-problems, see Aase [1985].



## 5. The existence of a stochastic equilibrium—general non-proportional treaties

### 5.1. *Introduction*

In this section we demonstrate the existence of a stochastic equilibrium as it is defined in Section 3.3. The market assumptions of Section 3 are in force. In particular there exists some probability measure $P^*$ equivalent to the originally given P such that any martingale y can be represented in the form of (3.21). When this holds true, the resulting partial equilibrium has the property that a small number of risks is sufficient to dynamically span the high dimensional space of all risks. Although markets are not complete at any given time, the random measure $\nu$ spans the space of all possible risks, proportional as well as nonproportional ones. We now turn to the first result in this section.

### 5.2. *The existence problem*

The following result can be proved:

**Proposition 5.1:** *Under Assumptions 2.1 and 3.1 the insurance economy* $IE_S$ = *(IE,* $\{F\}$, $L^2(\mathbf{x})$) *has a stochastic equilibrium with a Pareto optimal allocation.*

*Proof:* Let $(\pi, y_i;\ i \in I)$ be the static equilibrium for the insurance economy *IE* guaranteed by Proposition 2.1, where the premium functional $\pi(y) = E\{V(T)y(T)\} = E^*\{y(T)\}$, $y \in X$, and where $V(T)$ is the market marginal utility and $V(t) = E\{V(T) \mid F_t\}$ can be thought of as a market "spot price process of risk" at each time $t \in T$. Note that this result does not depend on any kind of dynamic spanning. Consider the $(P^*, F_t)$-martingales $w_i(t) = E^*\{y_i(T) \mid F_t\}$ for each $i \in I$. Clearly $w_i$ is optimal for agent i in the static economy, since $w_i(T) = y_i(T)$. By (3.21) there exist strategies $\mathbf{h}^{(i)} \in L^2(\mathbf{x})$ such that

$$w_i(t) = \int_{[0,t]} \iint_{R^I_-} \sum_{j \in I} h_j^{(i)}(s,\mathbf{u}) \cdot u_j \tilde{\nu}(d\mathbf{u};ds) \qquad i = 1, 2, \cdots, I,\ t \in T. \tag{5.1}$$

The budget constraint (3.27) is satisfied under P. By the projection theorem (5.1) can be replaced by $w_i(t) = x_i(0) + \int_0^t \mathbf{h}^{(i),S}(s) \cdot d\mathbf{x}(s) + m^{(i)}(t)$, where m is a $P^*$-martingale orthogonal to $\mathbf{x}$, $m^{(i)}(0) = 0$, and where $\mathbf{h}^{(i),S}$ is the associated proportional treaty closest to $\mathbf{h}^{(i)}$. By taking expectations in (5.1) under $P^*$ and evaluating at $t = T$, the terminal time, we get from this representation and the martingale property of $\mathbf{x}(t)$ under P* that $E^*\{w_i(T)\} = E^*\{y_i(T)\} = E^*\{x_i(0)\} = E^*\{x_i(T)\}$, which is the budget constraint (2.14) in the static economy. Thus, by using the reinsurance strategy $\mathbf{h}^{(i)}(t,\mathbf{u})$ and faced with a time zero portfolio $x_i(0)$ and market premiums $\pi(x_i) = \pi(y_i) = E^*\{y_i(T)\}$, insurer i can precisely obtain his optimal net reserve $y_i$ in the static insurance economy *IE*.

Suppose insurer i can obtain a strictly preferred portfolio $v_i >^i w_i$ by adopting a different reinsurance strategy $\mathbf{g}^{(i)} \in L^2(\mathbf{x})$. We call the associated unsystematic risk components $m_i^g(t)$. Then the market value $\pi$ must be strictly larger, i.e., $E^*\{v_i(T)\} > E^*\{w_i(T)\} = E^*\{y_i(T)\}$. Substituting the budget constraint for $v_i$, we get

$$E^*\{\sum_{j=0}^{I} g_j^{(i),S}(T, \mathbf{x})x_j(T) + m_i^g(T)\} > E^*\{y_i(T)\}$$

or,

$$E^*\{x_i(0) + \int_0^T \mathbf{g}^{(i),S}(s, \mathbf{x})d\mathbf{x}(s) + m_i^g(T)\} > E^*\{y_i(T)\},$$

i.e. $E^*\{x_i(T)\} > E^*\{y_i(T)\}$. But this contradicts the budget constraint (2.14) in the static reinsurance economy *IE*. Thus $(\mathbf{h}^{(i)}, w_i)$ is optimal for insurer i, and such a plan can be chosen for each $i \in I$. Suppose we replace the strategy $\mathbf{h}^{(I)}$ chosen in this way by

$$h_j^{*(I)}(t,\mathbf{u}) = 1 - \sum_{i=1}^{I-1} h_j^{(i)}(t,\mathbf{u}),\ j = 0, 1, 2, \cdots, I,\ t \in T,\ u \in R_+^I. \tag{5.2}$$

This is a permissible strategy since $\mathbf{h}^{*(I)} \in L^2(\mathbf{x})$ as well. Also $\mathbf{h}^{*(I)}$ generates the same optimal portfolio $w_I$ as does $\mathbf{h}^{(I)}$: To see this, first observe that from linearity of stochastic integrals

$$\sum_{j=1}^{I} \int_{[0,t]} \iint_{R_+^I} \left(1 - \sum_{i=1}^{I-1} h_j^{(i)}(s,\mathbf{u})\right) u_j \tilde{\nu}(d\mathbf{u};ds)$$

$$= \sum_{j=1}^{I} x_j(t) - \sum_{j=1}^{I} \int_{[0,t]} \sum_{i=1}^{I-1} h_j^{(i)}(s,\mathbf{u}) u_j \tilde{\nu}(d\mathbf{u};ds)$$

$$= \sum_{i=1}^{I} x_i(t) - \sum_{i=1}^{I-1} \int_{[0,t]} \sum_{j=1}^{I} h_j^{(i)}(s,\mathbf{u}) u_j \tilde{\nu}(d\mathbf{u};ds)$$

$$= \sum_{j=1}^{I} x_j(t) - \sum_{i=1}^{I-1} w_i(t).$$



Second, from the definition of $w_I(t)$ it follows that $\sum_{i=1}^{I} w_i(t) = \sum_{i=1}^{I} E^*\{y_i(T)|F_t\} = E^*\{\sum_{i=1}^{I} y_i(T)|F_t\} = E^*\{\sum_{i=1}^{I} x_i(T)|F_t\} = \sum_{i=1}^{I} E^*\{x_i(T)|F_t\} = \sum_{i=1}^{I} x_i(t)$ from market clearing in the static reinsurance economy *IE* and the martingale property of $\mathbf{x}(t)$ under P*. Hence $\sum_{i=1}^{I} x_i(t) - \sum_{i=1}^{I-1} w_i(t) = w_I(t)$, so the claim follows. This shows that $(\mathbf{h}^{*(I)}, w_I)$ is budget feasible for insurer I. Since $(\mathbf{h}^{(I)}, w_I)$ is optimal for I, so is $(\mathbf{h}^{*(I)}, w_I)$. From (5.2) and the above it follows that markets clear, so that (3.28–29) hold. Thus the conclusion of the proposition follows. □

Notice that we here have utilized the existence results in the static reinsurance economy *IE* to establish our result without the direct use of dynamic programming. This is particularly fortunate in our model due to the complexity of the Hamilton-Jacobi-Bellman equations in the present situation.

Similar implementation techniques as demonstrated in the above proof have earlier been developed within the context of financial economics (see Duffie and Huang [1985] and Cox and Huang [1991]).

## 5.3. *Expected utility*

As in Section 2.5 we now consider the case where preferences are represented by expected utility. Suppose each agent i wants to solve

$$\max U^i(y_i) \tag{5.3}$$

subject to $\pi(y_i) = \pi(x_i) = E\{V_T x_i(T)\}$, where $U^i(y_i) = E\{u_i(y_i(T))\}$ for some Bernoulli utility function $u_i$ with $u_i'(\cdot) > 0$ and $u_i''(\cdot) < 0$, $i \in I$.

**Theorem 5.1:** *Let Assumption 3.1 hold, suppose that $U^i(\cdot)$ is additively separable and regular $(u_i)$, and that $x_M(T)$ is strictly positive a.s. Then the insurance economy $IE_S = (IE, F, \mathrm{H}(\mathbf{x}))$ has a stochastic equilibrium with a set of Pareto optimal allocations $y_i(x_M(t))$, $i \in I$, $t \in T$, satisfying individual rationality. The premium functional is given by $\pi(y) = E\left\{V\left(x_M(T)\right) y\left(x_M(T)\right)\right\}$, where $V_T$ only depends on $x(T)$ through $x_M(T) = \sum_{i \in I} x_i(T)$. Furthermore, the premium at any time $t \in T$ of the remaining risk in* [t, T] *equals $\pi(y)(t) = (V(t))^{-1} E\left\{V(T)\left(y(T) - y(t-)\right) \mid \mathbf{F}_t\right\}$, where $V(t)$ only depends on $\mathbf{x}(t)$ through the term $x_M(t) = \sum_{i \in I} x_i(t)$.*

*Proof:* The proof essentially follows from the theory of this section, Proposition 5.1, and from Theorem 2.1. Here we use Proposition 5.1 and the saddle point theorem as follows: The results of this proposition hold under the conditions of Theorem 5.1, so that there exists an equlibrium. Define for strictly positive constant $k_1, k_2, \cdots, k_I$

$$\bar{U}(x_M) = \sup_{y \in X} \sum_{i \in I} k_i U^i(y_i(\mathbf{x})) \tag{5.4}$$

such that market clearing $\Sigma_i\ y_i(\mathbf{x}) = \Sigma_i\ x_i = x_M$ holds. Then we know from the saddle point theorem that $x_M$ is the solution to

$$\max_{x \in X_1} \bar{U}(x) \qquad \text{subject to } \pi(x) = \pi(x_M). \tag{5.5}$$

Problem (5.5) is equivalent to the problem

$$\max_{y \in X_1} E\{\bar{u}(y(T))\} \qquad \text{subject to } E\{V_T y(T)\} = E\{V_T x_M(T)\}, \tag{5.6}$$

where $\bar{u}(y) = \sup_{y_i} \Sigma_i k_i u_i(y_i)$ subject to $\Sigma_i y_i = y$. Since $u_i$ are strictly concave, increasing utility functions, so is $\bar{u}(\cdot)$. Also, since $x_M$ solves (5.6), there must exist some Lagrangian multipler $\lambda > 0$ in (5.6) such that $\frac{\partial \bar{u}}{\partial y}(x_M(T)) = \lambda V_T$, by the Euler equations. Thus $V_T = \lambda^{-1} \frac{\partial \bar{u}}{\partial y}(x_M(T))$ only depends on the vector $\mathbf{x}(T)$ through the sum of its components. The same story can be told at time $t < T$, i.e. $V_t = E\{V(x_T) \mid \mathcal{F}_t\}$ only depends on $\mathbf{x}(t)$ via $x_M(t) = \Sigma_i\ x_i(t)$. □

## 6. The term structure of interest rates

### 6.1. *Introduction*

In this section we develop an equilibrium model for the interest rate on risk free borrowing in our model. There is a well developed theory in financial economics for how the interest rate may result endogenous to the model (see e.g., Cox, Ingersoll and Ross [1985], Merton [1973a]). In the present case we need a careful examination of how uncertainty is revealed as time goes in our model, and this needs to be combined with the principles of economic equilibrium analysis and the associated stochastic calculus presented in sections 2–5 of this paper. Let us start by combining the premium formula (3.12) with Theorem 5.1. Then we get

$$\pi_t(y) = \frac{1}{V(x_M(t),t)} E\left\{V\Big(x_M(T),T\Big)\Big(y(T) - y(t\text{-})\Big) \mid \mathcal{F}_t\right\}, \qquad t \in \mathcal{T}. \tag{6.1}$$



By relaxing the assumption that $x_0(t) \equiv 1$ for all $t \in \mathcal{T}$, we still keep the above form of the premium functional, but the martingale property under P of $V(x_M(t),t)$ no longer automatically follows, neither is the expected value of $V(x_M(T),T)$ necessarily equal to unity any more.

### 6.2. *Equilibrium interest rate*

Suppose we define the equilibrium interest rate process, i.e., the interest rate on riskless short-time borrowing as follows: *It equals the stochastic process* $\{r(t), t \in \mathcal{T}\}$ *which is the return rate demanded by a security whose equilibrium price is always unity.* Recall our model for the net reserves in the market given in (4.4), and consider the market quantities $x_M(t)$, $a_M(t) = \Sigma_i\ a_i(t)$ and $z_M(t) = \Sigma_i\ z_i(t)$. Clearly the dynamic equation for the aggregated net reserves in the market $x_M(t)$ equals

$$x_M(t) = x_M(0) + \int_0^t \iint_{R^I_-} a_M(\mu v(a_M,s) - 1)\lambda^z_s F^z_s(d\mathbf{a})ds - \int_0^t \iint_{R^I_-} z_M \tilde{\nu}(d\mathbf{z};ds), \tag{6.2}$$

by linearity of stochastic integrals. Even if the expression for the market marginal utility in (3.23) is no longer valid, the regularity properties of $V(x_M(t),t)$ as a function of the arguments are inherited from the individual utility functions of the insurers, so that we can use the generalized Itô formula to obtain

$$\begin{aligned} V(x_M(t),t) = V(x_M(0),0) &+ \int_0^t \frac{\partial V}{\partial s}(x_M(s),s)ds \\ &+ \int_0^t \iint_{R^I_+} \frac{\partial V}{\partial x_M}(x_M(s),s)a_M\,(\mu v(a_M,s) - 1)\lambda^z_s F^z_s(d\mathbf{a})ds \\ &- \int_0^t \iint_{R^I_-} L^d(V,z_M)\lambda^z_s F^z_s(d\mathbf{z})ds \\ &- \int_0^t \iint_{R^I_-} \Big[V(x_M(s\text{-}) - z_M,s) - V(x_M(s\text{-}),s\text{-})\Big]\tilde{\nu}(d\mathbf{z};ds). \end{aligned} \tag{6.3}$$

where

$$L^d(V,z_M) = \left[V\left(x_M(s-) - z_M,s\right) - V\left(x_M(s-),s-\right) + \frac{\partial V}{\partial x_M}\left(x_M(s-),s\right)z_M\right], \quad (6.4)$$

and where $z_M$ in these expressions represents the jumps (if any) at time s of the aggregate market portfolio. Because the aggregate assets process $a_M$ is predictable, any jumps at time s in $x_M$ which are not known an instant before time s-, are entirely due to the aggregate claims process against the insurers, explaining the above notation. Using (6.1) and the results in Section 3.2, it follows that the real interest rate process $\{r(t), t \in \mathcal{T}\}$ on riskless short-time borrowing must satisfy

$$1 \equiv \frac{1}{V(x_M(t),t)} E\{\int_t^T V_s r(s)ds \mid F_t\},\ t \in \mathcal{T}, \quad (6.5)$$

or

$$V(x_M(t),t) = E\{\int_t^T V_s r(s)ds \mid F_t\}$$

$$= E\{\int_t^\tau V_s r(s)ds + V_\tau \mid F_t\},\ 0 \le t \le \tau \le T, \quad (6.6)$$

where the last equality follows by iterated expectations. By inserting (6.3), properly reformulated, into (6.6), we obtain the following

$$V(x_M(t),t) = E\left\{\int_t^\tau V_s r(s)ds + V_t + \int_t^\tau \frac{\partial V}{\partial s}\left(x_M(s),s\right)ds\right.$$
$$+ \int_t^\tau \iint_{R^l_+} \frac{\partial V}{\partial x_M}\left(x_M(s),s\right)a_M\left(\mu_s v(a_M,s) - 1\right)\lambda_s^\zeta F_s^\zeta(da)ds$$
$$- \int_t^\tau \iint_{R^l_+} L^d(V,z_M)\lambda_s^\zeta F_s^\zeta(dz)ds$$



$$\left. - \int_t^\tau \iint_{R^l_+} \left[V\left(x_M(s-) - z_M,s\right) - V\left(x_M(s-),s-\right)\right]\tilde{\nu}(dz;ds) \mid F_t\right\} \quad (6.7)$$

where $L^d(V,z_M)$ again is given in (6.4). Since the last term is a martingale difference, the conditional expected value, given $F_t$, equals zero for each $t \in \mathcal{T}$. Therefore

$$0 = E\left\{\int_t^\tau V_s r(s)ds + \int_t^\tau \frac{\partial V}{\partial s}\left(x_M(s),s\right)ds\right.$$
$$+ \int_t^\tau \iint_{R^l_+} \frac{\partial V}{\partial x_M}\left(x_M(s),s\right)a_M(\mu_s v(a_M,s) - 1)\lambda_s^\zeta F_s^\zeta(da)ds$$
$$\left. - \int_t^\tau \iint_{R^l_+} L^d(V,z_M)\lambda_s^\zeta F_s^\zeta(dz)ds \mid F_t\right\}. \quad (6.8)$$

Denoting the sum of all the "drift terms" above by $\int_t^\tau m(s)ds$, i.e. the sum of the three last terms inside the conditional expectation, we obtain the result that the equilibrium short-term real interest rate must satisfy

$$E\left\{\int_t^\tau \left(V_s r_s + m(s)\right)ds \mid F_t\right\} = 0, \qquad 0 \le t \le \tau \le T, \quad (6.9)$$

which implies that

$$r_t = -\frac{m(t)}{V_t}, \qquad t \in \mathcal{T}, \qquad \text{P-a.s.} \quad (6.10)$$

The right-hand side in (6.10) equals *minus the growth rate of the market's marginal utility process, or minus the rate of growth of the spot price of risk in the market*. Clearly, if at a certain time t the drift in the spot price of risk is positive, the real equilibrium interest rate $r_t$ is negative, and vice versa, a result which makes sound economic sense.

A few final remarks peculiar to the jump world can be given by considering (6.8–10): The right-hand side of (6.10) consists in reality of three terms. The first term can be interpreted as the time impatient rate $-\dfrac{\frac{\partial V}{\partial t}(x_M(t),t)}{V(x_M(t),t)}$ of the market.

The second term can be written $-\displaystyle\iint_{R^I_-}\left[\frac{V(x_M(t)-z_M,t)-V(x_M(t)}{V(x_M(t),t)(-z_M)}\,z_M\right]\lambda^z F^z_t(dz)$,

which can be interpreted as minus the expected value of the random intertemporal elasticity of substitution in aggregate claim sizes in the market at time t. These two terms together correspond to what is known as the subjective discount rate (of the representative agent, or the market as a whole) under full certainty. Finally the third term can be written $R_A(x_M(t),t)[a_M(t)(\lambda^a-\lambda^z)-Ez_M(t)]$, where $R_A$ represents the intertemporal absolute risk aversion coefficient inserted the aggregate portfolio in the market. This term reduces to $R_A(x_M(t),t)[-Ez_M(t)]$ under risk neutrality. Thus, the equilibrium real interest rate equals the analogous "subjective discount rate" plus this latter term. Notice that the presence of jumps has separated the relative risk aversion coefficient from the elasticity of substitution. In models with no jumps, such a separation has usually only been achieved by the introduction of non-separable preferences. For further discussions of these points in a financial economics framework, also related to the equity premium puzzle, see e.g., Aase [1992b].

In the case where the jump sizes are so small that higher than second order terms can be neglected, we obtain the limiting case that the riskfree rate equals the time impatience rate in the case when F and $\lambda$ are non-random (do not depend upon $\omega$).

### 6.3. *Discounting*

Consider the discounted marginal utility process in the market:

$$V_t^* = \exp\{-\int_0^t r_s ds\}V_t. \tag{6.11}$$

Then by Itô's lemma

$$dV_t^* = -r_t\exp\{-\int_0^t r_s ds\}V_t dt + \exp\{-\int_0^t r_s ds\}dV_t$$



$$= -\exp\{-\int_0^t r_s ds\}\iint_{R^I_+}\left[V\left(x_M(t-)-z_M,t\right) - V\left(x_M(t-),t-\right)\right]\tilde{v}(dz;dt), \tag{6.12}$$

where the last equality follows from (6.10) and (6.3). Thus $V_t^*$ is a $(P, \mathcal{F}_t)$-martingale. We are now back in the previous framework of sections 1–5, except from the fact that we still do not know whether or not $E(V_T^*)=1$. If it is, then (3.23) is again the correct representation for $V_t^*$, the discounted spot price process of risk. The premium of a risk y can now be written

$$\pi(y) = E\{V(T)y(T)\} = E\left\{\exp\{\int_0^T r_s ds\}\,V^*(T)y(T)\right\}. \tag{6.13}$$

Similarly, the equilibrium market premium of the remaining risk in [t, T] equals

$$\pi_t(y) = \frac{1}{V(t)}E\left\{V(T)\left(y(T)-y(t-)\right) \mid \mathcal{F}_t\right\}$$

$$= \frac{1}{\exp\{\int_0^t r_s ds\}\,V^*(t)}E\left\{\exp\{\int_0^T r_s ds\}\,V^*(T)\left(y(T)-y(t-)\right) \mid \mathcal{F}_t\right\},\ t\in\mathcal{T}. \tag{6.14}$$

In the special case where the process $\{r(t),\ t\in\mathcal{T}\}$ is deterministic, (6.14) simplifies to

$$\pi_t(y) = \exp\{\int_t^T r(s)ds\}E^*\left\{\left(y(T)-y(t-)\right) \mid \mathcal{F}_t\right\},\ t\in\mathcal{T}, \tag{6.15}$$

where $E^*$ is the expectation operator under $P^*$, and where the associated density process of this change of probability measure $\{V^*(t),\ t\in\mathcal{T}\}$ has a representation as given in (3.23). This follows, since from the above definition of r, the expression $E\left\{V^*(T)\exp\{\int_0^T r(s)ds\}\right\} = \pi(1) = E\left\{\exp\{\int_0^T r(s)ds\}\right\}$, which implies that $E\{V^*(T)\} = 1$ when r is deterministic.

## 7. Applications

### 7.1. *Introduction*

In the following we give some examples that fit into the theory of this paper. Consider the case where $\{\mathbf{z}(t), t \in T\}$ is an inhomogeneous, multivariate compound Poisson process. By this we mean that (see e.g., Snyder [1975]) (i) $\{N(t), t \in T\}$ is an inhomogeneous Poisson process with intensity function $\lambda(t) \geq 0$, $t \in T$, where $\lambda(t)$ is a real, non-random function; (ii) $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots$ is a sequence of mutually independent, identically distributed random vectors which are also independent of $\{N(t), t \in T\}$. The joint distribution of $\mathbf{z}^{(1)}$ is denoted by $F^z(\mathbf{z})$.

As a consequence of these assumptions the incomes process $\mathbf{a}(t)$ becomes a purely deterministic process. The process $\mathbf{z}(t)$ has the representation

$$\mathbf{z}(t) = \sum_{n=1} \mathbf{z}^{(n)} 1(\tau_n \leq t) = \sum_{n=1}^{N(t)} \mathbf{z}^{(n)}, \; t \in T. \tag{7.1}$$

Here $N(t)$ is the number of time instants in $(0, t]$ when accidents occur. Given that $\tau_n$ is the n'th time of accident, the claim sizes in the market $\mathbf{z}^{(n)}$ at this instant are distributed according to $F^z(\mathbf{z}) = P[z_1^{(n)} \leq z_1, \cdots, z_I^{(n)} \leq z_I]$, $\mathbf{z} \in R^I_+$, $n = 1, 2, \cdots$.

### 7.2. *A reinsurance version of the intertemporal CAPM*

As our first example let us assume that the market attitude towards claim size risk is represented by some time invariant, linear marginal dis-utility function, whereas the market is risk neutral in relation to frequency risk at each time instant. Thus $\mu(t) \equiv 1$ for all $t \in T$, and

$$v(z_M) = \alpha + \gamma z_M, \; \alpha, \gamma \text{ constants}, \tag{7.2}$$

$\alpha = 1 - \gamma E z_M$, $0 < \gamma \leq 1/E z_M$. Thus $v(z_M) = 1 + \gamma(z_M - E z_M) \geq 0$ P-a.s. so (3.18) of Assumption 3.1 is satisfied. The market marginal dis-utility function $V_T$ of (3.23) is here

$$V_T = \prod_{n=1}^{N(T)} v(z_M^{(n)}). \tag{7.3}$$

The market premium on any of the initial claims processes $\{z_i(t), t \in T\}$ can be computed as follows

$$\pi(z_i) = E\{V_T z_i(T)\} = E^*\{z_i(T)\}. \tag{7.4}$$



Using the last equality, we notice that since $\mathbf{z}(t)$ is a compound Poisson process under $P^*$ with local characteristics

$$(\lambda(t), F^*(d\mathbf{z}) = v(z_M)F^z(d\mathbf{z})). \tag{7.5}$$

it follows from properties of this process that

$$\pi(z_i) = E^*(z_i^{(1)}) \left( \int_0^T \lambda(t)dt \right). \tag{7.6}$$

The expectation under $P^*$ we compute as follows:

$$E^*(z_i^{(1)}) = \int\int_{R^I_+} z_i F^*(d\mathbf{z}) = \int\int_{R^I_+} z_i \Big(1 + \gamma(z_M - E(z_M)\Big) F^z(d\mathbf{z})$$

$$= E(z_i^{(1)}) + \gamma \operatorname{cov}(z_i^{(1)}, z_M^{(1)}), \tag{7.7}$$

where $z_M^{(1)} = z_1^{(1)} + \cdots + z_I^{(1)}$. The same expression holds for any of the risks $z_j$, $j \neq i$, and from linearity of the premium functional $\pi(\cdot)$ it follows that

$$\pi(z_M) = \left( \int_0^T \lambda(t)dt \right) \left( E(z_M^{(1)}) + \gamma \operatorname{var}(z_M^{(1)}) \right). \tag{7.8}$$

Rearranging gives

$$\pi(z_i) - E\{z_i(T)\} = \frac{\operatorname{cov}(z_i^{(1)}, z_M^{(1)})}{\operatorname{var}(z_M^{(1)})} \Big( \pi(z_M) - E\{z_M(T)\} \Big), \; i \in I. \tag{7.9}$$

The above expression is recognized as a version of the intertemporal capital asset pricing model, *where the risk's "beta" is the normalized covariance between any of its claim sizes and the corresponding accumulated claim size in the market.*

The above derivation could alternatively be carried out using the first of the equalities in (7.4) and the expression for $V_T$ given above in (7.3). In the present case this calculation is straightforward. In other situations it can sometimes simplify the computations considerably to use the measure $P^*$ and known properties of the process $\{\mathbf{z}(t), t \in T\}$ under this measure.

Finally notice that the intertemporal CAPM derived above does not follow from arbitrage arguments alone: In the case with linear marginal dis-utility given in (7.2), $v(\mathbf{z}) = v(z_M)$ follows from Theorem 5.1, which is an equilibrium result.

### 7.3. *Logarithmic dis-utility*

Suppose that

$$v(z_M) = \frac{\alpha}{\beta - \alpha z_M}, \text{ where } \beta > \alpha z_M \text{ P-a.s., } \alpha, \beta > 0,$$

$$\alpha^{-1} = \iint_{R^I_+} \frac{1}{\beta - \alpha z_M} F^z(\mathbf{dz}). \quad (7.10)$$

The market premium of any of the initial claims processes can be computed using P*. The result is

$$\begin{aligned}
\pi(z_i) &= E^*\{z_i(T)\} \\
&= \left(\int_0^T \lambda(t)dt\right) E^*\{z_i^{(1)}\} \\
&= \left(\int_0^T \lambda(t)dt\right) \iint_{R^I_+} \frac{z_i \alpha}{\beta - \alpha z_M} F^z(\mathbf{dz}) \\
&= \left(\int_0^T \lambda(t)dt\right) \frac{1}{\iint_{R^I_-} \frac{F^z(\mathbf{dz})}{\beta - \alpha z_M}} \iint_{R^I_-} \frac{z_i}{\beta - \alpha z_M} F^z(\mathbf{dz}) \\
&= \left(\int_0^T \lambda(t)dt\right) \alpha E\left\{z_i^{(1)} \frac{1}{\beta - \alpha z_M^{(1)}}\right\}.
\end{aligned} \quad (7.11)$$

By the linearity of the premium functional,

$$\pi\{z_M\} = \left(\int_0^T \lambda(t)dt\right) \alpha E\left\{z_M^{(1)} \frac{1}{\beta - \alpha z_M^{(1)})}\right\}. \quad (7.12)$$

Rearranging, we obtain

$$\pi\{z_i\} = \frac{\pi(z_M)}{E\{z_M^{(1)}/(\beta - \alpha z_M^{(1)})\}} E\left\{\frac{z_i^{(1)}}{\beta - \alpha z_M^{(1)}}\right\}, \ i \in I. \quad (7.13)$$



The risk premium $\pi\{z_i\} - E(z_i(T))$ is finally seen to be

$$\pi\{z_i\} - E(z_i(T)) = \frac{\operatorname{cov}(z_i^{(1)}, \frac{1}{\beta - \alpha z_M^{(1)}})}{\operatorname{cov}(z_M^{(1)}, \frac{1}{\beta - \alpha z_M^{(1)}})} \left\{\pi(z_M) - E(z_M(t)\right\}, \quad (7.14)$$

where we have used that $E\{v(z_M^{(1)}\} = 1$.

According to the theory of Section 5.3, if preferences are represented by expected logarithmic utility, the above model results. The constants $\alpha$ and $\beta$ may be derived from the corresponding parameters of the individual utility functions together with the constants $k_i$ of (5.6). These are in its turn determined by the budget constraints and $F^z(\mathbf{dz})$.

### 7.4. *Power dis-utility*

Suppose that

$$v(z_M) = \frac{(\gamma - z_M)^\alpha}{\kappa}, \ \alpha \in (-1, 0), \quad (7.15)$$

where the constant $\gamma$ and the distribution of $z_M$ satisfies $(\gamma - z_M) \geq 0$ P-a.s., and where $\kappa = E(\gamma - z_M)^\alpha$. Here $Ev(z_M) = 1$, so this function is permitted and Assumption 3.1 holds. The premium of $z_i$ is computed as follows:

$$\begin{aligned}
\pi(z_i) = E^*\{z_i(T)\} &= \left(\int_0^T \lambda(t)dt\right) E^*\{z_i^{(1)}\} \\
&= \left(\int_0^T \lambda(t)dt\right) E\left\{\frac{(\gamma - z_M^{(1)})^\alpha}{\kappa} z_i^{(1)}\right\}.
\end{aligned} \quad (7.16)$$

Proceeding as before we obtain the following expression for the risk premium of any of the initially underwritten risks $z_i$:

$$\pi(z_i) - E\{z_i(T)\} = \frac{\operatorname{cov}(z_i^{(1)}, (\gamma - z_M^{(1)})^\alpha)}{\operatorname{cov}(z_M^{(1)}, (\gamma - z_M^{(1)})^\alpha)} \left(\pi(z_M) - E(z_M(T)\right). \quad (7.17)$$

The function $v(\cdot)$ above is clearly motivated from individual power utility theory. This is a particularly interesting case, since power utility does not give rise to optimal sharing rules which are linear, in which case optimal risk sharing is of the non-proportional type.

### 7.5. *Other examples*

Several similar computations can be carried out, for example to compute the premium of a stop loss contract on insurer i's initially underwritten risk. Such illustrations are presented elsewhere (Aase [1992]). Similarly other preferences can be studied, e.g. $v(z_M) = \exp\{\alpha z_M - \ln(ee^{\alpha z_M})\}$, where $\alpha > 0$ is some constant. This corresponds to the Esscher model of actuarial science introduced by Bühlmann [1980]. In our framework this model corresponds to preferences represented by expected exponential utility, where the constant $\alpha$ may be derived from the individual parameters, together with the budget constraints and $F^{\zeta}(dz)$. Some computations for this model are presented in Aase [1992].

In the above examples the market attitude towards frequency risk could be introduced by letting the function $\mu(t) \geq 0$ differ from the constant 1. In the simple case where the market is risk neutral with regard to claim sizes, the risk premium of $z_i$ equals

$$\pi(z_i) - E\{z_i(T)\} = \left(\int_0^T \lambda(t)(\mu(t) - 1)dt\right)E\{z_i^{(1)}\}. \tag{7.18}$$

It is seen from (7.18) that the risk premium is non-negative in the case where $\mu(t) \geq 1$ for all $t \in T$. This corresponds to the insurers setting their premiums in the market according to a larger frequency of accidents ($= \mu(t)\lambda(t)$) than they expect to be the true frequency ($= \lambda(t)$). $P^*$ has here the effect of risk adjusting the joint frequency of claims in the market by a time dependent loading function $\mu(t)$ on frequency, ultimately determined by the market.

## 8. Summary

Given the many different lines of reinsurance, our results can be interpreted roughly as follows: In some lines the uncertainty about frequency is crucial, but claim sizes given that claims have occurred are not equally important. Within non-life insurance auto insurance is an example. This is due to the dramatic effects weather conditions can have on claim numbers, such as icy roads in the winter time, wet summers etc., whereas repair costs are much easier to predict, given that the claim numbers are known.



In principle is traditional life-insurance another example, disregarding unit-linked products. Formally we do not cover life insurance/pension plans in the model of this section, but the basic idea follows from the general theory of this paper. Here the claim sizes are written into the contracts, so only uncertainty regarding the time points of deaths remains. The insurers usually have reliable mortality statistics, so that the probability distributions, and thus the "forces of mortality" functions, are known. In principle risk neutrality ought to follow from competition and the strong law of large numbers, since the number of similar policies that each insurer holds is usually large. In reality the companies use some kind of loading on the force of mortality, so that the term $\mu$ is larger than one for some t and age groups, and the life insurers are risk averse in regard to mortality. Similarly, for pension plans $\mu$ is smaller than one, since overestimating the life lengths of the pensioners now gives the right safety margins for risk averse insurers.

In other lines of reinsurance, like marine or oil, claim sizes are considered to be the most important. This is because relatively few events happen, but given an accident the damage cost may take on a whole range of values, including very large ones, so that reinsurance is here a necessity. The market concentrates its attitude towards risk to claim size distributions in this kind of business, since basically only one accident can happen per contract, and free reinstatements are naturally excluded from such contracts.

In yet other lines both sources of uncertainty may be judged to be important and cause the reinsurers to behave with risk aversion jointly on both frequency and conditional claim sizes. This is perhaps the most natural situation, and one obvious example is unit-linked life insurance (if stock market risk is modeled by random measures instead of the usual diffusion processes).

In the paper we have shown that any general risky treaty can be spanned by the random measure $\tilde{\nu}$. Moreover, any such risk can be decomposed into a proportional treaty and a general, non-proportional treaty orthogonal to this one. Within this framework we have demonstrated the existence of a stochastic equilibrium, and we have outlined how the unique proportional component of an optimal strategy can be found by dynamic programming. The model for the term structure of the interest rate in this jump type model demonstrates some interesting differences from the analogous continuous type models. In particular it does not require as strong smoothness properties on preferences as does the diffusion based analysis. Also the jumps turn out to separate the intertemporal elasticity of substitution from the coefficient of relative risk aversion, thus achieving much the same as non-separable preferences does in continuous type models.

As our applications indicate, the different attitudes towards risk are quite naturally modeled within the present framework. We observe quite generally that in reinsurance the market premium of a risk typically depends upon:

(i) The stochastic properties of the risk itself, here represented by the marginal local characteristics $(\lambda(t), F^{\zeta}(dz))$.

(ii) The stochastic relationship between the particular risk and claims in the market as a whole, described by the joint distribution $F'(dz)$ and the common intensity function $\lambda(t)$ of the claims in the market.

(iii) The market attitude towards risk, represented by the market marginal dis-utility $V_T$, which in our model can be summarized by essentially two components: the market marginal dis-utility function $v(z_M)$ on claim sizes, and the market's marginal attitude towards frequency risk represented by the function $\mu(t)$.

(iv) The total reserves $a_M$ of all the insurers in the market (according to the remark made in section 3.5).

This last point has been frequently observed to be of importance in reinsurance markets, where the total capacity allocated to a certain line of insurance clearly affect premiums.

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