ON THE PREMIUM OF EQUITY-LINKED INSURANCE CONTRACTS

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Abstract

We will deal with the valuation of equity-linked insurance contracts such as unit links. We will introduce a premium principle based on the optimization of expectation bounded and coherent measures of risk. The premium principle seems to present some interesting properties. Indeed, firstly, it is sub-additive and favors diversification. Secondly, it integrates both actuarial and financial risks, and does not have to impose independence between them. Thirdly, it provides the insurer with hedging strategies. Finally, it is very easy to use in practice since one only has to solve linear programming problems, despite the fact that risk measures are not linear at all.

Key words. Risk measure, Premium, Equity-linked contract.

J.E.L. Classification, G22, G23, G12.

Sobre la prima de contratos de seguro ligados al mercado financiero

Resumen

Estudiaremos el problema de la valoración de contratos de seguro ligados al mercado financiero, tales como las anualidades o rentas ligadas a índices bursátiles. Introduciremos un principio de prima basado en la optimización de medidas de riesgo coherentes y acotadas por la media. Este principio parece presentar una serie de propiedades de interés. En efecto, en primer lugar, es sub-aditivo, por lo que favorece la diversificación. Segundo, se integran los riesgos actuariales y financieros, y no hace falta suponer independencia de los mismos. Tercero, se proporcionarán estrategias de cobertura para el asegurador. Y cuarto, la prima del contrato es fácil de calcular en las aplicaciones prácticas, puesto que sólo hay que resolver

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problemas de programación lineal, pese a que las medidas de riesgo están lejos de ser lineales.

Palabras clave. Medidas de riesgo, Prima, Contrato ligado al mercado financiero.

Clasificación J.E.L., G22, G23, G12**.**

I. Introduction

Artzner et al. (1999) introduced the axioms and properties of their "coherent measures of risk" and later many authors extended the discussion (for example, Rockafellar *et al*., 2006, introduced the "expectation bounded measures of risk", and Brown and Sim, 2009, defined the "satisfying measures").

Since then many actuarial and financial problems have been revisited. For instance, with respect to purely actuarial topics, *Gao et al*. (2007) deal with equilibrium prices, *Kaluszka* (2005), *Bernar and Tian* (2009) or *Centeno and Simoes* (2009) study optimal reinsurance problems and *Barbarin and Devolder* (2005) or *Gaillardetz* (2008) focus on equity linked annuities. Mixed (*i.e*., both actuarial and financial) problems are presented in *Wang* (2000), *Hamada and Sherris* (2003), or *Balbás et al.* (2008), amongst others, and pure financial problems may be found in *Föllmer and Leukert* (2000), *Nakano* (2003), *Nakano* (2004), *Staum* (2004), etc.

There are several reasons justifying this growing interest in new risk measures, but two of them may deserve special attention. Firstly, if asymmetric returns are involved then the classical standard deviation is not compatible with the Second Order Stochastic Dominance and the usual Utility Functions (Ogryczak and Ruszczynski, 1999 and 2002). Secondly, but also very importantly, modern risk measures may be understood as possible capital losses and capital requirements, which provides us with information that is not yielded by the standard deviation.

This article focuses on the valuation of insurance products that are linked with the financial market (*i.e.*, with the evolution of some risky asset) with special attention to the risk level given by a general expectation bounded and coherent risk measure. As will be indicated, there are many potential products involved, though "the unit links" are probably the most popular contracts. Besides, expectation bounded and coherent risk measures are general enough and contain many important particular cases (the *Dual Power Transform* of Wang, 2000, the *Conditional Value at Risk* of Rockafellar *et al*., 2006, etc.).

The paper's outline is as follows. The second section will be devoted to presenting the general framework we are going to deal with. In Section 3 we will draw on an original idea of *Balbás et al.* (2010*b*) so as to introduce a new Premium Principle for contracts affected by both actuarial and financial risks. Actually, the paper above gives a general method that allows us to extend pricing rules in incomplete financial markets by minimizing risk measures, while we adapt that pricing rule to the concrete problem we are studying. The introduced premium principle seems to present interesting properties. Indeed, firstly, it is sub-additive and favors diversification. Secondly, it integrates both actuarial and financial risks, and does not have to impose independence between them. Thirdly, it provides the insurer with hedging strategies that make the global risk faced by the insurer vanish.

The fourth section will be devoted to analyzing models represented by a discrete probability space. This special setting is important for two reasons. Firstly, it may significantly simplify computations in practical studies, and secondly, it is not restrictive at all since for every real situation there exist discrete approximations as close as desired. Concrete equity-linked insurance products and other applications are given in Section 5, with special focus on the usual equity-linked annuities. They are modeled according to an original idea of *Balbás et al.* (2010*a*).

The last section of the paper summarizes the most important conclusions.

II. Preliminaries and notations

Let as assume that $t = 0$ and $t = T$ represent the current and a future date respectively. Consider the probability space $(\Omega, \Im, \mu)$ composed of the set Ω (states of nature or states of the world), the σ-algebra $\Im$ (information available at $t = T$) and the probability measure μ. Suppose that $\Omega = \Omega_a \times \Omega_f$ contains both those states of nature belonging to Ω_a and related to the evolution of a set of insurance policies, and the states of nature belonging to Ω_f, related to the evolution of a financial market. Similarly, representing with the symbol $\otimes$ the tensor product of σ-algebras, we will assume that $\Im = \Im_a \otimes \Im_f$, but we will not impose independence between the actuarial and the financial risk. In other words, if μ_a and μ_f denote the natural (marginal) projections of the probability measure μ on Ω_a and Ω_a respectively, in general, we will not assume the fulfillment of the equality $\mu = \mu_a \otimes \mu_f$, $\mu_a \otimes \mu_f$ denoting the usual tensor product of μ_a and μ_f.[2]

Let be $p \in [1,2]$ and suppose that $L^p(\Omega, \Im, \mu)$ (henceforth L^p for short) denotes the usual space of $\Im -$ measurable random variables y such that the expectation of $\left| y^p \right|$ is finite. Denote by $q \in [2, \infty]$ the conjugate of p ($1/p + 1/q = 1$). It is well known that the Riesz Representation Theorem states that L^q is the dual space of L^p (Horváth, 1966, or Luenberger, 1969). In particular, every real valued linear and continuous function on L^p takes the form

$$L^p \ni y \to E\left(q^* y\right) \in \Re,$$

$q^* \in L^q$ being an arbitrary element that only depends on the linear function we are dealing with, and $E(-)$ denoting the mathematical expectation of any random variable.
Analogous ideas and notations will apply for the actuarial and the financial problems, *i.e.*, L^p (L^q) will be used for the joint problem, while $L^p(\Omega_a)$ and $L^p(\Omega_f)$ ($L^q(\Omega_a)$ and $L^q(\Omega_f)$) will represent the actuarial and the financial ones respectively. With the usual convention we can assume that $L^p(\Omega_a) \subset L^p$ and $L^p(\Omega_f) \subset L^p$, and identical inclusions hold if *q* plays the role of *p*.

Next let us introduce the risk measurement criterion and the pricing rule of the financial market. Consider a general risk function

$$\rho : L^p \to \Re$$

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[2] Actually, the independence between both probability spaces is not a restrictive assumption, but some authors have pointed out that the results of some insurance companies may affect the behaviour of some financial markets. For instance, significant sales from pension funds could worsen the performance of some index. Thus, we will never impose independence (see Gaillardetz, 2008, for an alternative discussion).

Since $L^p \supset L^2$, we will deal with risk measures that may be extended beyond L^2. Denote by

$$\Delta_\rho = \left\{ z \in L^q : -E(yz) \leq \rho(y)\cdot.\forall y \in L^p \right\}. \tag{1}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma\left(L^q, L^p\right)-$ compact,[3] $z \geq 0$ and $E(z) = 1$ for every $z \in \Delta_\rho$, and

$$\rho(y) = Max\left\{ - E(yz) : z \in \Delta_\rho \right\} \tag{2}$$

holds for every $y \in L^p$. Summarizing, we have:

Assumption 1. *The set Δ_ρ given by (1) is convex and $\sigma\left(L^q, L^p\right)-$ compact, its elements are non-negative and have an expected value equal to one, and (2) holds for every $y \in L^p$.* □

The assumption above is closely related to the Representation Theorem of Risk Measures stated in *Rockafellar et al*. (2006). Following their ideas, it is easy to prove that the fulfillment of Assumption 1 holds if and only if ρ is continuous and:

Translation invariant, i.e., $\rho(y+k) = \rho(y) - k$, for every $y \in L^p$ and $k \in \Re$.
Sub-additive, i.e., $\rho(y + y') = \rho(y) + \rho(y')$, for every $y, y' \in L^p$.
Homogeneous, i.e., $\rho(\lambda y + k) = \lambda \rho(y)$, for every $y \in L^p$ and $\lambda > 0$.
Mean dominating, i.e., $\rho(y) \geq -E(y)$, for every $y \in L^p$.
Decreasing, i.e., $\rho(y) \leq \rho(y')$, for every $y, y' \in L^p$ with $y \geq y'$.

According to *Artzner et al*. (1999) and *Rockafellar et al*. (2006), risk measures satisfying the properties above (or Assumption 1) are called Coherent and Expectation Bounded. Particular interesting examples are the *Conditional Value at Risk* (*CVaR*) of *Rockafellar et al.* (2006), the *Weighted Conditional Value at Risk* (*WCVaR*) of *Cherny* (2006), the *Compatible Value*

[3] See *Horvàth* (1966) or *Luenberger* (1969) for further details about $\sigma\left(L^q, L^p\right)-$ compact sets.

at Risk (*CCVaR*) of *Balbás and Balbás* (2009), the Dual Power Transform (*DPT*) of *Wang* (2000) and the Wang Measure (Wang, 2000), among many others.

If ρ satisfies the properties above, then so do the restrictions of ρ to $L^p(\Omega_a)$ and $L^p(\Omega_f)$. Thus Assumption 1 still holds if Δ_ρ is replaced by the set $\Delta_{\rho,a}$ (or $\Delta_{\rho,f}$) below and $y \in L^p(\Omega_a)$($y \in L^p(\Omega_f)$),

$$\Delta_{\rho,a} = \left\{z \in L^q(\Omega_a) : -E(yz) \le \rho(y) \because \forall y \in L^p(\Omega_a)\right\}$$
$$\Delta_{\rho,f} = \left\{z \in L^q(\Omega_f) : -E(yz) \le \rho(y) \because \forall y \in L^p(\Omega_f)\right\}$$

With respect to the financial market, we will assume that it is perfect and complete, that is, there are no transaction costs or other imperfections/frictions and every final pay-off $y \in L^2(\Omega_f)$ may be reached at T by means of a self-financing portfolio adapted to the arrival of information (see Cochrane, 2001, for further details about the usual assumptions of a pricing model in finance). Actually, the completeness of the market is not necessary for most of the results we are going to deal with, but it significantly simplifies the exposition, and most of the classical pricing models (binomial, Black and Scholes, Heston, etc) are complete. Accordingly, we can go beyond the Riesz Representation Theorem above. Indeed, in order to prevent the existence of arbitrage (Cochrane, 2001), there is a unique Stochastic Discount Factor $z_\pi \in L^2(\Omega_f)$ such that $z_\pi > 0$ almost surely and

$$\Pi(y) = e^{-rT} E(yz_\pi) \tag{3}$$

holds for every $y \in L^2(\Omega_f)$, r denoting the riskless interest rate and $\Pi(y)$ denoting the initial (at $t=0$) price of every final pay-off y. If one takes the riskless asset $y = e^{rT}$ then (3) obviously implies that

$$E(z_\pi) = 1 \tag{4}$$

III. Pricing equity-linked insurance contracts

Consider an insurance contract whose final value (at *T*) depends of both the actuarial and the financial risk. Denote by $g \in L^2$ the random amount that the insurer will pay to her/his client. Then, the insurer may look for "protection" in the financial market so as to minimize the global risk of her/his net position. Though the financial market is complete, it would be obviously a very restrictive assumption to consider that so is the insurance (or the joint) market, so the insurer is pricing *g* in an incomplete market. There are several approaches dealing with the valuation with risk measures of contingent claims in an incomplete market (Wang, 1999, Hamada and Sherris, 2003, Nakano, 2003 and 2004, etc.), though we will follow a minor modification of that of *Balbás et al.* (2010*b*). Accordingly, the premium that at *t=0* the insurer will receive for *g* is given the optimal value of the minimization problem

$$\begin{cases} Min \quad \rho(y-g)+P \\ \quad E(yz_\pi) \leq P \\ \quad P \in \Re \\ \quad y \in L^2(\Omega_f) \end{cases} \tag{5}$$

(P, y) being the decision variable. The interpretation of (5) is clear. Indeed, according to the first constraint, P represents the value (at *T*) of the hedging strategy that the insurer will use so as to compensate possible capital losses provoked by *–g*, [4] whereas $\rho(y-g)$ is (the value at *T* of) the capital requirement or reserve that the insurer must add so as to prevent severe damages and/or negative evolutions of the financial market. Thus, the objective $\rho(y-g)+P$ reflects (the value at *T* of) the capital needed by the insurer so as to sell the risk *g* and hedge the position, and therefore, according to *Balbás et al.* (2010*b*), this is (the value at *T* of) the price that the insurer must receive for *g*. Notice that the risk level that the insurer has to face vanishes, since for the solution *y* of (5) one has that

$$\rho(y-g+(\rho(y-g)))=0 \tag{6}$$

[4] The client will receive the amount *g*, so the insurer will receive *-g*.

because ρ is translation invariant, and the amount $\rho(y-g)$ has been paid by the customer as the first part of the global premium.
Once (5) has been solved, its optimal value must be multiplied by the discount factor e^{-rT} to compute the insurance premium, since it is paid at *t=0*.

Actually, there is a minor difference between (5) and the optimization problem proposed in *Balbás et al.* (2010*b*), since these authors only deal with a financial problem and do not incorporate any actuarial risk. However, straightforward modifications of their arguments allow us to adapt their major results about duality and Lagrange multipliers for (5). Thus, since (5) is obviously a feasible mathematical programming problem, we will give without proof the following theorem.

Theorem 2. *Suppose that (5) is bounded.*[5] *Consider the dual problem*

$$\begin{cases} Max & E(gz) \\ & z \in \Delta_\rho \\ & z - z_\pi \in L^2(\Omega_f)^\perp \end{cases} \tag{7}$$

z being the decision variable and $L^\perp$ *denoting the orthogonal manifold of every subspace* L *of* L^2*. Then (7) is solvable and its maximum equals the primal infimum of (5). Furthermore,* (y^*, P^*) *and* z^* *solve (5) and (7) respectively if and only if the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} E(gz) - E(y^* z) \le E(gz^*) - E(y^* z^*), \quad \forall z \in \Delta_\rho \\ E(z_\pi y^*) = P^* \\ z^* - z_\pi \in L^2(\Omega_f)^\perp \\ y^* \in L^2(\Omega_f), P^* \in \Re, z^* \in \Delta_\rho \end{cases} \tag{8}$$

hold. □

[5] Hereafter we will assume that (5) is bounded. Actually, there are some "pathological" situations leading to unbounded problems, but they must be overcome with appropriate modifications of the risk measure ρ. Further details may be found in *Balbás and Balbás* (2009) and *Balbás et al.* (2010*b*). Anyway, it is worth pointing out that the feasible set of (7) does not depend on *g*, so (5) becomes unbounded for every *g* if so is for some particular risk, for instance the null one.

Bearing in mind (1) and (4), it is easy to see that $\left(y^* - P^*, 0\right)$ and z^* satisfy (8) if and only if so do $\left(y^*, P^*\right)$ and z^*, which implies that one can always look for a solution making the variable P vanish. Thus, the unknown P^* may be removed in System (8), which is illustrated in the following result.

Corollary 3. *If (5) is solvable then there exists a solution* $\left(y^*, P^*\right)$ *such that* $P^* = 0$. *Moreover* $\left(y^*, 0\right)$ *and* z^* *solve (5) and (7) if and only if*

$$\begin{cases} E(gz) - E\left(y^* z\right) \leq E\left(gz^*\right), \quad \forall z \in \Delta_\rho \\ E\left(z_\pi y^*\right) = 0 \\ z^* - z_\pi \in L^2\left(\Omega_f\right)^\perp \\ y^* \in L^2\left(\Omega_f\right), z^* \in \Delta_\rho \end{cases} \tag{9}$$

hold.

Proof. As pointed out above, we can assume that the primal solution $\left(y^*, P^*\right)$ satisfies $P^* = 0$. Hence, the second and fourth expressions in (9) trivially follow from the equivalent expressions in (8), while the first one becomes obvious if we prove that $E(y^* z^*) = 0$. Bearing in mind the second condition in (9) and the properties $z_\pi - z^* \in L^2\left(\Omega_f\right)^\perp$ and $y^* \in L^2\left(\Omega_f\right)$, we have that $E(y^* z^*) = E\left(y^* z_\pi\right) = 0$. □

The introduced price of g makes sense even if g does not depend on the actuarial risk ($g \in L^2\left(\Omega_f\right)$). Let us see that we are not modifying its price in such a case.

Corollary 4. *If* $g \in L^2\left(\Omega_f\right)$ then the optimal value of (5) and (8) equals $\Pi(g)e^{rT}$.

Proof. If z^* is the dual solution then $z^* - z_\pi \in L^2\left(\Omega_f\right)^\perp$. Thus, taking into account (3),

$\Pi(g)e^{rT} = E\left(z_\pi g\right) = E\left(z^* g\right)$. □

The later corollary justifies that the price of g will be denoted by $\Pi(g)$ in what follows.

Every premium principle should be sub-additive, since otherwise customers would prefer to sign several contracts rather than a single one. Moreover, the seminal paper by *Deprez and Gerber* (1985) already justified that most of the usual premium principles are given by convex functions. Let us show that our premium respects these requirements.

Corollary 5. *If* $g, g_1, g_2 \in L^2$ *and* $\alpha \geq 0$ *then* $\Pi(g_1 + g_2) \leq \Pi(g_1) + \Pi(g_2)$ *and* $\Pi(\alpha g) = \alpha \Pi(g)$.

Proof. According to Theorem 2 there exists z^* (8)-feasible such that

$$\Pi(g_1 + g_2) = e^{-rT} E\left(z^*(g_1 + g_2)\right) = e^{-rT} E\left(z^* g_1\right) + e^{-rT} E\left(z^* g_2\right) \leq \Pi(g_1) + \Pi(g_2)$$

where the last inequality also follows from Theorem 2. The second expression may be proved in a similar manner. □

Finally, let us point out that condition $z^* - z_\pi \in L^2(\Omega_f)^\perp$ may be given in a different manner, which trivially implies that (7), (8) and (9) may be accordingly modified and the new results remain true. We will draw on usual notations so as to represent mathematical conditional expectations.

Proposition 6. *Suppose that* $z^* \in \Delta_\rho$. *Then,* $z^* - z_\pi \in L^2(\Omega_f)^\perp$ *holds if and only if*

$$E\left(z^* / \omega_f \in A\right) = E\left(z_\pi / \omega_f \in A\right) \tag{10}$$

for every set $A \in \Im_f$ *with* $\mu_f(A) > 0$.

Remark. *If it is not confusing Expression (9) will simplify to*

$$E\left(z^* / z_\pi\right) = z_\pi \tag{11}$$

since this new notation is much more intuitive.

Proof of Proposition 6. Suppose that $z^* - z_\pi \in L^2(\Omega_f)^\perp$, $A \in \Im_f$ and $\mu_f(A) > 0$. Since the market is complete the indicator function 1_A is a reachable pay-off,[6] so $z^* - z_\pi \in L^2(\Omega_f)^\perp$ leads to

$$\iint_{\Omega_a \times A} (z^* - z_\pi) d\mu = 0 .$$

Hence,

$$\int_{\Omega_a \times A} z^* d\mu = \int_A z_\pi d\mu_f \ , \tag{12}$$

which trivially leads to (10).

Conversely, suppose that (10) holds. Then, (12) also holds for every $A \in \Im_f$ with $\mu_f(A) > 0$, and therefore

$$\int_\Omega z^* y d\mu = \int_{\Omega_f} z_\pi y d\mu_f \tag{13}$$

for every $y \in L^2(\Omega_f)$, because (13) holds for every simple random variable in $L^2(\Omega_f)$ and the set of simple random variables is dense in this space. □

IV. The discrete case

This section will be devoted to showing that (7) and (9) make it possible to solve (5) in practice, *i.e.*, one can easily compute a "fair price" for Risk *g* as well as the optimal hedging strategy for its financial risk. For illustrative reasons we are going to deal with a discrete probability space $(\Omega, \Im, \mu)$, though it is also possible to solve the problem in the general framework.[7]

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[6] Recall that $1_A(\omega) = 1$ if $\omega \in A$ and $1_A(\omega) = 0$ otherwise.

[7] Actually, in *Balbás et al.* (2009) a more complicated optimal reinsurance problem is solved in a "continuous" probability space, and *Anderson and Nash* (1987) present very complete information about algorithms related to infinite-dimensional linear optimization problems. However, we will deal with discrete spaces here to simplify the exposition. Moreover, discrete probability spaces have been often used in actuarial and financial approaches (Nakano, 2003, Calafiore, 2007, Mansini *et al.*, 2007, Gaillardetz, 2008, etc.), since they permit us to give accurate approximations of every probability space.

If we consider that $\Omega_a = \{\omega_1^a, \omega_2^a, ..., \omega_n^a\}$ and $\Omega_f = \{\omega_1^f, \omega_2^f, ..., \omega_m^f\}$ then, according to Proposition 6, Problem (7) becomes

$$\begin{cases} Max & \sum_{i=1}^{n}\sum_{j=1}^{m} g_{i,j} z_{i,j} \\ & \left(z_{i,j}\right)_{i,j} \in \Delta_\rho \\ & \sum_{i=1}^{n} \mu_{i,j} z_{i,j} = \mu_{f,j} z_{\pi,j}, \quad j = 1,2,...,m \end{cases} \tag{14}$$

$\left(z_{i,j}\right)_{i=1,j=1}^{i=n,j=m}$ being the decision variable. Obviously, we are denoting $g_{i,j} = g\left(\omega_i^a, \omega_j^f\right)$, $z_{i,j} = z\left(\omega_i^a, \omega_j^f\right)$, $\mu_{i,j} = \mu\left(\omega_i^a, \omega_j^f\right)$, $z_{\pi,j} = z_\pi\left(\omega_j^f\right)$ and $\mu_{f,j} = \mu_f\left(\omega_j^f\right)$, *i=1,2,…,n* and *j=1,2,….,m*.

Analogously, the Karush, Kuhn Tucker like conditions (9) become in this case

$$\begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{m} g_{i,j} z_{i,j} - \sum_{j=1}^{m}\left(\sum_{i=1}^{n} z_{i,j} y_j^*\right) \le \sum_{i=1}^{n}\sum_{j=1}^{m} g_{i,j} z_{i,j}^*, \quad \forall \left(z_{i,j}\right)_{i,j} \in \Delta_\rho \\ \left(z_{i,j}\right)_{i,j} \in \Delta_\rho \\ \sum_{i=1}^{n} \mu_{i,j} z_{i,j} = \mu_{f,j} z_{\pi,j}, \qquad j = 1,2,...,m \\ \sum_{j=1}^{m} z_{\pi,j} y_j^* = 0 \end{cases}$$

(15)

$\left(z_{i,j}^*\right)_{i=1,j=1}^{i=n,j=m}$ and $\left(y_j^*\right)_{j=1}^{j=m}$ being the unknowns. Notice that (15) may be easily solved in practice if the solution $\left(z_{i,j}^*\right)_{i=1,j=1}^{i=n,j=m}$ is known, because in such a case $\left(y_j^*\right)_{j=1}^{j=m}$ becomes the only unknown of the system. To compute $\left(z_{i,j}^*\right)_{i=1,\,j=1}^{i=n,\,j=m}$ one must solve Problem (14), but this is frequently a linear optimization problem that may be easily solved by the popular simplex method. For instance, a very important particular risk measure satisfying the linearity of (14) is the *Conditional Value at Risk* or *CVaR*. This risk measure is becoming very interesting for both researchers and practitioners.[8]

According to *Rockafellar et al*. (2006), if $0 < \upsilon < 1$ denotes the confidence level of the *CVaR* (henceforth we will denote $CVaR_\upsilon$, if necessary) then we have that the sub-gradient given in (1) becomes

$$\Delta_{CVaR_\upsilon} = \left\{ z \in L^\infty : 0 \le z \le \frac{1}{1-\upsilon}, E(z) = 1 \right\}.$$

Thus, bearing in mind that we are dealing with discrete spaces, (14) becomes

$$\begin{cases} Max & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} g_{i,j} z_{i,j} \\ & 0 \le \left(z_{i,j}\right)_{i,j} \le \dfrac{1}{\upsilon}, \qquad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,m \\ & \displaystyle\sum_{i=1}^{n} \mu_{i,j} z_{i,j} = \mu_{f,j} z_{\pi,j}, \quad j = 1,2,\ldots,m \end{cases}$$

(16)

Notice that constraint

$$E(z) = 1$$

does not have to be imposed. Indeed, it trivially follows from the last restriction of (16) and Expression (4).

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[8] The linearity of (14) also holds for important closely related risk measures such as the *WCVaR* (Cherny, 2006) and the *CCVaR* (Balbás and Balbás, 2009), among others.

V. Examples and applications

Maybe the most important practical application is given by the equity linked annuities or unit links. Actually these products have already been studied with risk measures beyond the variance, like *VaR* and *CVaR* (Barbarin and Devolder. 2005, Gaillardetz., 2008, etc.) though we will propose a quite different approach more closely related to the discussion above. Actually we will follow the initial approach of *Balbás et al.* (2010*a*), where it is shown that the optimization of modern risk measures may be used so as to compute the loading rate of these products. With respect to *Barbarin and Devolder* (2005), *Gaillardetz* (2008), and other interesting contributions, our analysis makes the global risk of the insurer vanish (see (6)). Moreover, as already said, we do not have to impose independence between both the actuarial and the financial risk. Finally, but also very important, we do not have to impose any concrete pricing model in the financial market, i.e., every complete arbitrage free pricing model may be used

Suppose for instance that *T=1* year is an initial horizon and consider *k* clients of the insurer. The j^{th}-client will pay the premium P_j at *t=0* and will receive the pay-off

$$g_j = \begin{cases} H_j, & not - alive \\ aP_j I, & alive \end{cases} \qquad (17)$$

where $0 < a \le 1$ and *I* denotes the (annual) realized return of a chosen financial asset (an index, usually). Alternative modifications of (17) may be considered according to the specific properties of the contract. [9] With the notation of previous sections we have that

$$g = \sum_{j=1}^{k} g_j \ . \qquad (18)$$

Actually we could price every individual contract rather than the global portfolio of policies represented in (18), but the sub-additivity of the premium principle Π (see Corollary 5) justifies that pricing the global

[9] For instance, one can assume that there is a guaranteed minimum amount if the customer survives, and therefore the pay-off becomes $Max\{g_j, C\}$, where g_j is given by (17).

portfolio leads to cheaper and more competitive products without facing higher levels of risk (see (6)).

Once $\Pi(g)$ has been computed by using (7) and (9) (or (15) and (16)) the global loading rate must be divided so as to calculate the loading rate of every particular policy. We will not discuss this second part which is beyond our focus. Nevertheless, classical actuarial methods (probably related to mortality tables) may apply.

Though equity linked annuities are very important equity linked insurance contracts, it is worth pointing out that the analysis of this paper may apply for alternative insurance policies. For instance, an illustrative example may be a *bonus-malus* system that links the lack of claims and the financial market, [10] or a policy involving the hole integrated wealth of the customer, composed of both her/his goods and her/his assets. According to the sub-additivity of the premium principle Π (Corollary 5) the integrated treatment will improve the global insurance price.

For all of these possible contracts the developed theory applies and minimizes both the (global, actuarial and financial) risk of the insurer and the cost of the contract.

VI. Conclusions

Modern coherent and expectation bounded measures of risk have been used in many actuarial and financial problems, and pricing issues are a very important particular case. The approach of this paper deals with the valuation of equity-linked insurance contracts such as equity linked annuities and other products. We have proposed a premium principle based on the optimization of expectation bounded and coherent risk measures that seems to present several interesting properties. Indeed, firstly, it is sub-additive and convex, and therefore it favors diversifications. Secondly, it integrates both actuarial and financial risks, and does not have to impose independence between them. As pointed out by several authors, the absence of independence might be a restrictive assumption in some applications. Third, it provides the insurer with hedging strategies, since the global risk of the insurance company vanishes. Finally, it is very easy to use in practice since

[10] *i.e.*, the premium reduction is not only related to the number of claims of the policy, but also with the evolution of some financial market or security.

one only has to solve linear programming problems, despite the fact that risk measures are not linear at all.

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