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LETTERS

An elementary derivation of Hattendorff's theorem

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Abstract

For a general fully continuous life insurance model, the variance of the loss-at-issue random variable is the expectation of the square of the discounted value of the net amount at risk at the moment of death. In 1964 Jim Hickman gave an elementary and elegant derivation of this result by the method of integration by parts. One might expect that the method of summation by parts could be used to treat the fully discrete case. However, there are two difficulties. The summation-by-parts formula involves shifting an index, making it somewhat unwieldy. In the fully discrete case, the variance of the loss-at-issue random variable is more complicated; it is the expectation of the square of the discounted value of the net amount at risk at the end of the year of death times a survival probability factor. The purpose of this note is to show that one can indeed use the method of summation by parts to find the variance of the loss-at-issue random variable for a fully discrete life insurance policy.

Keywords Loss at issue · Hattendorff's theorem · Summation by parts · Net amount at risk

The celebrated Hattendorff Theorem [6] in life contingencies is perhaps best viewed as an application of the result that increments of a martingale over disjoint time intervals are uncorrelated [3]. The purpose of this note is to present an elementary derivation of a version of the theorem. The word "elementary" is used in the sense that the key tool in the derivation is the technique of summation by parts.

We consider the model, presented in Sect. 7.4 of [1] and also in Sect. 5.5 of [4], of a general fully discrete life insurance on (x). For $j = 1, 2, 3, \ldots$, the death benefit in the jth policy year is b_j, payable at time j, which is the end of the policy

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year; the premium paid in the jth policy year is π_{j-1}, payable at time $j-1$, which is the beginning of the policy year. Let K_x denote the curtate future lifetime of (x); for simplicity, we shall use K for K_x. The *insurer's loss at issue* random variable is

$$L := v^{K+1} b_{K+1} - \sum_{j=0}^{K} v^j \pi_j \tag{1}$$

For $j=0, 1, 2, \ldots$, let ${}_jV$ denote the reserve of the policy at time j; also, define

$$\rho(j) := v^{j+1}(b_{j+1} - {}_{j+1}V),$$

the present value of the *net amount at risk* at the end of policy year $j+1$. The version of Hattendorff's Theorem that we shall derive is

$$\mathrm{Var}[L] = \mathrm{E}[[\rho(K)]^2 \times p_{x+K}]. \tag{2}$$

Note that we do not necessarily assume ${}_0V=0$.

Derivation: It follows from the reserve recursion formula,

$${}_jV + \pi_j = v\left({}_{j+1}V p_{x+j} + b_{j+1} q_{x+j}\right),$$

that

$$v^j \pi_j = \Delta(v^j {}_jV) + \rho(j) q_{x+j}, \tag{3}$$

where Δ denotes the *forward difference operator*. Applying (3) to Eq. (1) yields

$$L = v^{K+1} b_{K+1} - \sum_{j=0}^{K} [\Delta(v^j {}_jV) + \rho(j) q_{x+j}]. \tag{4}$$

Because

$$\sum_{j=0}^{K} \Delta(v^j {}_jV) = v^{K+1} {}_{K+1}V - v^0 {}_0V = v^{K+1} {}_{K+1}V - \mathrm{E}[L],$$

Eq. (4) can be rewritten as

$$\begin{aligned} L - \mathrm{E}[L] &= \rho(K) - \sum_{j=0}^{K} \rho(j) q_{x+j} \\ &= \rho(K) - \varphi(K), \end{aligned} \tag{5}$$

with the definition

$$\varphi(k) := \sum_{j=0}^{k} \rho(j) q_{x+j}, \quad k = 0, 1, 2, \ldots \tag{6}$$

By (5),

$$\begin{aligned}\mathrm{Var}[L] &= \mathrm{E}[[\rho(K) - \varphi(K)]^2] \\ &= \mathrm{E}[[\rho(K)]^2] + E[[\varphi(K)]^2] - 2E[\rho(K)\varphi(K)].\end{aligned}$$

Thus, deriving formula (2) is equivalent to showing

$$\mathrm{E}[[\varphi(K)]^2] = 2\mathrm{E}[\rho(K)\varphi(K)] - \mathrm{E}[[\rho(K)]^2 q_{x+K}]. \tag{7}$$

Because $\Pr[K = k] = {}_kp_x - {}_{k+1}p_x = -\Delta_k p_x$,

$$\mathrm{E}[[\varphi(K)]^2] = -\sum_{k=0}^{\infty} [\varphi(k)]^2 \Delta_k p_x. \tag{8}$$

To evaluate (8), we use the summation-by-parts formula,

$$\sum_{k=m}^{n} g(k)\Delta h(k) = g(k)h(k)\big|_{k=m}^{k=n+1} - \sum_{k=m}^{n} h(k+1)\Delta g(k).$$

Hence,

$$\begin{aligned}\mathrm{E}[[\varphi(K)]^2] &= -[\varphi(k)]^2 {}_kp_x\Big|_{k=0}^{k=\infty} + \sum_{k=0}^{\infty} {}_{k+1}p_x\, \Delta\big([\varphi(k)]^2\big) \\ &= [\varphi(0)]^2 {}_0p_x + \sum_{k=0}^{\infty} {}_{k+1}p_x\big([\varphi(k+1)]^2 - [\varphi(k)]^2\big).\end{aligned} \tag{9}$$

From (6),

$$\varphi(k) = \varphi(k+1) - \rho(k+1)q_{x+k+1}, \quad k = 0, 1, 2, \ldots,$$

which implies

$$[\varphi(k+1)]^2 - [\varphi(k)]^2 = 2\varphi(k+1)\rho(k+1)q_{x+k+1} - [\rho(k+1)q_{x+k+1}]^2.$$

With ${}_{k+1}p_x \times q_{x+k+1} = \Pr[K = k+1]$, the series on the right-hand side of Eq. (9) is

$$\begin{aligned}&\sum_{k=0}^{\infty} \big([\varphi(k+1)]^2 - [\varphi(k)]^2\big)_{k+1}p_x \\ &\quad = \sum_{k=0}^{\infty} 2\varphi(k+1)\rho(k+1)\Pr[K = k+1] - \sum_{k=0}^{\infty} [\rho(k+1)]^2 q_{x+k+1} \Pr[K = k+1] \\ &\quad = 2\{\mathrm{E}[\varphi(K)\rho(K)] - \varphi(0)\rho(0)q_x\} - \{\mathrm{E}[[\rho(K)]^2 q_{x+K}] - [\rho(0)]^2(q_x)^2\} \\ &\quad = 2\mathrm{E}[\varphi(K)\rho(K)] - \mathrm{E}[[\rho(K)]^2 q_{x+K}] - [\varphi(0)]^2\end{aligned} \tag{10}$$

because $\rho(0)q_x = \varphi(0)$. It follows from (9) and (10) that we have derived (7). Thus we have presented an elementary derivation of formula (2).

Remarks (i) We were motivated to seek this summation-by-parts derivation because there is a rather straightforward integration-by-parts derivation in the fully continuous case [5, 7]. Let T_x denote the future lifetime of (x). The fully continuous analogues of (1) and (2) are

$$L = v^{T_x} b_{T_x} - \int_0^{T_x} v^t \pi_t \mathrm{d}t \tag{11}$$

and

$$\mathrm{Var}[L] = \mathrm{E}[[v^{T_x}(b_{T_x} - {}_{T_x}V)]^2], \tag{12}$$

respectively. To derive (12), we apply the following form of *Thiele's differential equation*,

$$v^t \pi_t \mathrm{d}t = \mathrm{d}(v^t {}_tV) + v^t(b_t - {}_tV)\mu_{x+t}\mathrm{d}t,$$

to (11), yielding

$$L = v^{T_x} b_{T_x} - \int_0^{T_x} \mathrm{d}(v^t {}_tV) - \int_0^{T_x} v^t(b_t - {}_tV)\mu_{x+t}\mathrm{d}t.$$

Because $\int_0^{T_x} \mathrm{d}(v^t {}_tV) = v^{T_x} {}_{T_x}V - v^0 {}_0V = v^{T_x} {}_{T_x}V - \mathrm{E}[L]$, we obtain

$$L - \mathrm{E}[L] = v^{T_x}(b_{T_x} - {}_{T_x}V) - \int_0^{T_x} v^t(b_t - {}_tV)\mu_{x+t}\mathrm{d}t. \tag{13}$$

Hence (12) is proved if we can show that the expectation of the square of the right-hand side of (13) is

$$\mathrm{E}[[v^{T_x}(b_{T_x} - {}_{T_x}V)]^2].$$

This is equivalent to showing

$$\mathrm{E}[[\int_0^{T_x} v^t(b_t - {}_tV)\mu_{x+t}\mathrm{d}t]^2] = 2\mathrm{E}[v^{T_x}(b_{T_x} - {}_{T_x}V) \times \int_0^{T_x} v^t(b_t - {}_tV)\mu_{x+t}\mathrm{d}t]. \tag{14}$$

Equation (14), simpler than its discrete analogue (7), can be readily verified by an integration by parts, as shown on page 43 of [5].

(ii) One may better understand (5) by noting that $\rho(j)q_{x+j}$ is the present value of the cost of insurance based upon the net amount at risk for policy year $(j+1)$.

(iii) Formula (2) is particularly useful if the death benefit, payable at the end of the year of death, is a face amount plus the reserve, because the net amount at risk is then just the face amount. (The face amount can be allowed to change from year to year.) Type B Universal Life insurance policies have such death benefits [2, 8].

(iv) As noted above, we do not necessarily assume ${}_0V = 0$. For $j = 0, 1, 2, \ldots$, let

$$ {}_jL := v^{K_{x+j}+1} b_{j+K_{x+j}+1} - \sum_{k=0}^{K_{x+j}} v^k \pi_{j+k} $$

be the time-j prospective loss random variable; this is (7.4.4) in [1]. Then,

$$ \mathrm{E}[{}_jL] = {}_jV; $$

see (7.4.5) in [1]. Formula (2) is generalized as

$$ \mathrm{Var}[{}_jL] = \mathrm{E}\Big[[v^{K_{x+j}+1}(b_{j+K_{x+j}+1} - {}_{j+K_{x+j}+1}V)]^2 \times p_{x+j+K_{x+j}} \Big]. $$

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