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IMPLICIT LOADINGS IN LIFE INSURANCE RATEMAKING AND COHERENT RISK MEASURES

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ABSTRACT

In this paper we study a premium calculation principle applied to life insurance based on a coherent risk measure called Proportional Hazard Transform. This is based on a probability distortion by means of Wang's distortion power function (Wang S. (1995)). We will see how this theoretically supports the practical use of implicit loadings in life insurance ratemaking. Either life insurance contracts with death coverage or life annuities are considered as exemplifications.

KEY WORDS:
Implicit loading, distortion function, hazards risk transform, coherent risk measure, life insurance.

RESUMEN

En este artículo se propone un principio de cálculo de primas para seguros de vida, basado en la medida de riesgo coherente, esperanza distorsionada con la función de distorsión de Wang (1995) en forma de potencia, la denominada "Proportional Hazards Transforms" (PH). Este principio propone una justificación teórica a la práctica habitual de recargar de manera implícita las probabilidades de fallecimiento y supervivencia. Se considera como ejemplo la modalidad de seguro de vida con cobertura de fallecimiento, el seguro vida

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PALABRAS CLAVE: Recargo implícito, función de distorsión, transformada del tanto instantáneo, medida de riesgo coherente, seguro de vida.

1. INTRODUCTION

In order to protect themselves against the risk arising from fluctuations in claims severities, insurance companies calculate security loadings. These may be explicitly calculated or implicitly assumed in the ratemaking, the later being the most common practice in life insurance.

When considering life insurance with death coverage, an adverse claim severity consists in lives lasting shorter than expected. Thus a common practice when calculating premiums is adding an implicit loading either defined as a safety margin conceived as a percentage of the death probabilities q_x , or using a mortality table with higher probabilities than those of the human group taken into account. The last is equivalent to consider a higher mortality instantaneous rate (for instance considering outdated mortality tables).

The opposite situation is found in life insurance with survival benefits, where an adverse claim severity occurs when life last longer than expected. In this case the insurance company can implicitly load the premium using mortality tables with lower death probabilities that decreases the instantaneous mortality rate.

Wang (1995) proposes a loaded premium calculation for non-life insurance using distorted probabilities. It is called proportional hazard transform (PH), and is in fact a coherent risk measure (Artzner, P. (1999)) with a distortion function of the form $g(u) = u^{1/\rho}$, where it is shown that the necessary condition for such a risk measure to be coherent is $\rho \geq 1$.

This article applies this result in the two following ways.

The more immediate one consists in applying the result from Wang (1996) to the case of survival insurance (i.e. life annuity insurance) showing that the common practice of implicitly loading the premium considering a lower instantaneous mortality rate can be considered in fact as a proper coherent risk measure. This is worked out expressing the net premium with respect to the survival function, then applying the proportional hazard transform (which stands in the case $\rho \geq 1$ and is coherent) and checking that this corresponds to a new instantaneous mortality rate proportional to the former by a factor of $1/\rho$. This step is in fact done in a second stage (see section 5).

But the first stage and main way of the study (see section 4), is the case of death coverage products (i.e. whole life insurance contracts) which is in fact out of Wang's result. This is motivated by the fact that considering an implicit loading would be equivalent to the calculation of a net premium corresponding to a higher instantaneous mortality rate. If we tried then to follow the same scheme as in the survival products case, we would find the obstacle that trying to express the net premium with respect to the survival function would no longer reward us with an expression that could be considered as a proportional hazard transform. A careful inspection of the expression obtained through this method will tell us that it is in fact a function of the proportional hazard transform this time with an exponent $\rho \in (0,1)$. Our goal consists in showing that this function still is a coherent risk measure.

Therefore this article seeks to justify a standard practice in life insurance showing that it may be viewed as the application of a coherent risk measure.

Section 2 is recalls some basic concepts on risk measures and premium calculation principles, while the proportional hazard transform is introduced along section 3. We will consider whole life insurance and life annuity contracts as exemplifications.

2. RISK MEASURES BASED ON PREMIUM CALCULATION PRINCIPLES

Premium calculation principles can be considered candidates for being risk measures. Although there is a consensus about how to calculate the net premium, there are many ways to aggregate the loading in order to obtain a premium reflecting the risk accepted by the insurer. On the other hand, risk measures are very interesting because they allow to quantify the way the premium compensates the insurer for the risk associated to the loss.

Before introducing a premium based on a risk measure, let us remember a formal definition of the later and the properties a risk measure must fulfill to be coherent (Artzner, P. (1999)).

Definition 1. Given a loss, modeled by means of a non-negative random variable X, a risk measure $H(X)$ is a functional $H: X \to \mathbb{R}$. Artzner (1999) suggests the following four axioms for a risk measure to be considered coherent:

I. Translation Invariance. For any random variable X and any non-negative constant b,

$$H(X+b)=H(X)+b.$$

This axiom states that if the loss X were increased by a fixed amount b, the risk would increase by the same amount.

II. Subadditivity. For all pairs of random variables X and Y,

$$H(X+Y)\leq H(X)+H(Y).$$

This axiom states that an insurance company cannot reduce its risk by dividing its business in smaller blocks.

III. Positive homogeneity. For any random variable X and any non-negative constant a>0

$$H(aX)=a\,H(X).$$

This axiom states that a change in the monetary unit does not change the risk measure.

IV. Monotonicity. For every pair of random variables X and Y, such that $P_r(X\leq Y)=1$ then

$$H(X)\leq H(Y).$$

This axiom states that if one risk loss is not greater than another for all states of nature, the risk measure of the former cannot be greater than the risk measure of the later.

For instance it is very well known that the risk measure based on the expected value principle is

$$H(X)=(1+\alpha)\mu_X \quad (\alpha\geq 0),$$

where α is the risk loading and μ_X stand for the mean of the loss X. When $\alpha = 0$, the risk measure is called pure or net premium. Other important premium principles are for instance, the variance and the standard deviation:

$$H(X) = \mu_X + \beta\sigma_X^2 \qquad \beta \geq 0$$
$$H(X) = \mu_X + \gamma\sigma_X \qquad \gamma \geq 0.$$

It is known (Yiu-Kuen Tse (2009) p.118) that these premium calculation principles are not coherent risk measures since the expected value principle does not verify the axiom of invariance under translations, and the standard deviation principle does not verify the monotony axiom. On the other hand, the net premium principle ($\alpha = 0$) is a coherent risk measure.

3-THE PROPORTIONAL TRANSFORM OF THE INSTANTANEOUS RATE AS A RISK MEASURE.

Consider a risk $X \geq 0$ with distribution function and survival function:

$$\begin{aligned} F(x) &= \Pr\left(X \leq x\right) \\ S(x) &= 1 - F(x). \end{aligned} \tag{1}$$

The net premium based on the expected value risk measure expressed by means of the survival function is:

$$E\left(X\right) = \int_0^\infty x\, dF(x) = \int_0^\infty S(x)dx.$$

We must now define a loaded premium likely to be better adjusted to the risk and based on the distortion function (Wang(1995)):

Definition 2: Given a risk X with survival function S(x), and a non-decreasing function $g:[0,1] \to [0,1]$ with $g(0) = 0,\ g(1) = 1$ called the distortion function, the risk premium adjusted to the distorted probability risk measure is:

$$E_g\left(X\right) = \int_0^\infty g\left(S(x)\right) dx. \tag{2}$$

Supposing that g and S have first derivatives, the distortion function verifies the following properties (Wang 1996):

1. g(S(x)) is non-decreasing .
2. $0 \leq g(S(x)) \leq 1$ for any $x \in [0, +\infty)$.
3. If g and S are continuous functions, g(S(x)) can be considered as the survival function of another random variable Y with density function given by
   $$f_Y(x) = -\frac{d\,g(S(x))}{dx} = -g'(S(x))S'(x) = g'(S(x))f(x).$$
   Therefore g'(S(x)) is a weighting function of the density function f(x). Moreover, if g is concave then
   $$\frac{d\,g'[S(x)]}{dx} = g''[S(x)]S'(x) \geq 0.$$

Therefore the distortion function allows us to define a new random variable Y, since g(S(x)) has the properties of a survival function.

We now consider a power function as a special case of distortion (Yiu-Kuen Tse. (2009), p.129). This case is well known in non-life insurance because it satisfies the properties of a coherent risk measure. It is also well known that considering a power of the survival function results in a proportional instantaneous mortality rate. As we are going to see soon, this has an interesting interpretation in life insurance.

Definition 3: A proportional transform of the instantaneous rate is a Wang measure with the following distortion function:

$$g(u) = u^{1/\rho},\ \rho > 0. \tag{3}$$

In this case a new random variable Y is defined from the original X, with survival function and premium adjusted to the risk given by:

$$\begin{gathered} S_Y(x) = \left(S(x)\right)^{1/\rho}\ ,\ \rho > 0 \\ \Pi_\rho(X) = E(Y) = \int_0^{+\infty} \left(S(x)\right)^{1/\rho} dx. \end{gathered} \tag{4}$$

We can deduce from definition 3 the following consequences:

1. E(Y) is an increasing function with respect to ρ. The higher is ρ, the higher will be the risk-adjusted premium. Thus ρ can be considered as a risk aversion parameter (see Yiu-Kuen Tse (2009) p.129).

2. The instantaneous rates of the random variables X and Y are proportional.

Recalling that:

$$S_Y(t) = \left(S(t)\right)^{1/\rho} = \left(\exp\left(-\int_0^t \mu_X(u)\,du\right)\right)^{1/\rho} = \exp\left(-\int_0^t \frac{1}{\rho}\mu_X(u)\,du\right).$$

We can write:

$$\mu_Y(t) = \frac{1}{\rho}\ \mu_X(t), \quad \rho > 0\ , \ \ t \geq 0. \tag{5}$$

Therefore the X and Y instantaneous rates of mortality are proportional. The new random variable Y is called the proportional instantaneous rate transform of X with parameter ρ (Wang (1996)).

This transform only requires that ρ > 0, though in the context of general insurance ρ ≥ 1 is considered in order to give more weight to the tail of the risk distribution.

Assuming that

$$S_Y(x) = \left(S(x)\right)^{1/\rho} \quad , \ \rho \geq 1,$$

the Y survival function decreases slower than the X one, with greater probabilities for larger values of the variable, so the risk-adjusted premium or the loaded premium verifies $\Pi_\rho(Y) = E\left(Y\right) \geq E\left(X\right)$, the difference being the security loading.

As indicated in (Wang (1995)), the risk-adjusted premium reflects the risk of the original loss, and the decision maker's risk aversion is adjusted by means of the parameter values ρ.

As shown in (Wang (1995)), for ρ ≥ 1 the distorted risk probabilities are a coherent risk measure. In fact, the constraint ρ ≥ 1 is mainly necessary for the measure to fulfill the subadditivity property, the other axioms requiring only ρ >0.

4. NET PREMIUM CALCULATION FOR A WHOLE LIFE INSURANCE CONTRACT.

Whole life insurance is a death coverage insurance such that the insurer undertakes to pay the guaranteed policy benefit to the beneficiaries, whatever the moment of the insured's death could be (see for example Bowers et al. (1997) p.94). It is an insurance policy with a fixed amount and random maturity.

Let us consider a whole life insurance contract for an individual (x) in continuous time, with one monetary unit as the insured capital. In this case the risk can be modeled by the random variable T(x), residual life or time remaining until the insured's death. The following assumptions hold:

1. At the time of death a monetary unit is paid.
2. i is the technical rate of interest.
3. A newborn's death time is a continuous random variable X, with survival function S(x). Then the random variable T(x) has a distribution function $G_x(t)$ and a survival function $S_x(t)$ whose expressions depending on S(x) are given by (Bowers et al (1997) p.52) :

$$\begin{aligned} G_x(t) &= \frac{S(x)-S(x+t)}{S(x)} = 1-\frac{S(x+t)}{S(x)} \\ S_x(t) &= 1-G_x(t) = \frac{S(x+t)}{S(x)} \ . \end{aligned} \tag{6}$$

4. Defining $v=(1+i)^{-1}$, the loss associated to the policy is then defined by means of the random variable:

$$Z = v^{T(x)} \tag{7}$$

Applying the actuarial equivalence principle, the following expression is obtained (Bowers et al. (1997) p.95) for the pure premium:

$$\Pi(Z) = \int_0^{+\infty} v^t \, dG_x(t) \tag{8}$$

In order to adapt the actuarial equivalence principle to premium calculation based on the distortion function, this is now expressed depending on $S_x(t)$.

$$\Pi(Z)=\int_0^{+\infty} v^t\, dG_x(t)=-\int_0^{+\infty} v^t\, dS_x(t)\,. \qquad (9)$$

Changing the variable to $v^t = z$:

$$\Pi(Z)=\int_0^1 z\, dS_x\left(\frac{\ln z}{\ln v}\right)\,.$$

Integrating by parts:

$$z = u \qquad dz = du$$

$$dS_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)=dv \qquad v=S_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)$$

$$\Pi(Z)=\left(z\, S_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)\right)_0^1-\int_0^1 S_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)dz=\left(1\ \ S_x(0)\right)-\int_0^1 S_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)dz=1-\int_0^1 S_x\left(\frac{\mathrm{Ln}\, z}{\mathrm{Ln}\, v}\right)dz$$

We finally obtain an expression for the pure premium based on $S_x(t)$

$$\Pi(Z)=1-\int_0^1 S_x\left(\frac{\ln z}{\ln v}\right)dz\,. \qquad (10)$$

Now a loaded premium is obtained substituting the survival function into the power distortion function. In (11) we find its expression, where the subscript ρ emphasizes the dependency of the premium on the previous choice of the power function:

$$\Pi_\rho(Z)=1-\int_0^1\left(S_x\left(\frac{\ln z}{\ln v}\right)\right)^{1/\rho} dz. \qquad (11)$$

It is clear that the exponent should be $\rho \le 1$ for the loaded premium to be greater than the pure premium.

We are now going to show how this loaded premium can be deduced considering the proportional transformation of the instantaneous rate, and also that it is a coherent measure of risk (remember that in this case $\rho \leq 1$).

Theorem 1. The loaded premium (11) equals the pure premium of another random variable with the same survival law model, but with instantaneous mortality rate proportional to the one of the random variable X, by a proportionality factor of 1/ρ.
Proof:

In fact, if we integrate by parts expression (11) and writing $r = \frac{1}{\rho}$:

$$u = \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^r \qquad du = r \frac{1}{z \log v} S'_x \left(\frac{\ln z}{\ln v} \right) \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^{r-1} dz$$

$$dv = dz \qquad\qquad v = z$$

$$\Pi_\rho(Z) = 1 - \int_0^1 \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^r dz =$$

$$= 1 - z \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^r \Bigg|_0^1 + \int_0^1 z r \frac{1}{z \log v} S'_x \left(\frac{\ln z}{\ln v} \right) \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^{r-1} dz = \int_0^1 z\, d \left(S_x \left(\frac{\ln z}{\ln v} \right) \right)^r .$$

Now changing variable $z = v^t$:

$$\Pi_\rho(Z) = -\int_0^\infty v^t d\left(S_x(t)\right)^r = -\int_0^\infty v^t d\left(S_x(t)\right)^{1/\rho} . \qquad (12)$$

Comparing (12) with (9), we see that it corresponds to the pure premium of an insurance of the same kind, though for a new random variable Y with survival function $S_Y(t) = \left(S_x(t)\right)^{1/\rho}$. Therefore we are in the same situation described in (5):

$$\mu_Y(t) = \frac{1}{\rho} \mu_X(t), \qquad (13)$$

where $\mu_X(t)$ is now the instantaneous rate of the original variable X. **(Q.E.D.)**

Thus we can conclude that for any survival law the loaded premium coincides with the pure premium that would be obtained for that law, though with a proportional rate by a factor $\frac{1}{\rho} \geq 1$. Therefore the new instantaneous rate is higher and this represents an adverse claim experience for the insurer.

Going now into de first step mentioned during the introduction, the following theorem shows that the subadditivity property is also verified by the premium defined in (10):

Theorem 2: For every pair of non-negative random variables U and V, and the risk measure given by:

$$H_\rho(U) = 1 - \int_0^1 \left(S_U(z)\right)^{1/\rho}\, dz, \quad 0 < \rho \leq 1 \ , z = v^t \ , \quad v = \frac{1}{1+i},$$

the subadditivity property holds:

$$H_\rho(U+V) \leq H_\rho(U) + H_\rho(V). \tag{14}$$

We will proceed quite similarly as is done in Wang (1996). Let us firstly show a previous lemma which plays a similar role as the one that can be found in Wang (1996).

Lemma 1. If $0 < a < b$ and $\rho \leq 1$ then $\forall x \geq 0$ the following holds:

$$\left(-(x+b)^{1/\rho}\right) - \left(-(x+a)^{1/\rho}\right) \leq \left(-b^{1/\rho}\right) - \left(-a^{1/\rho}\right).$$

Proof:

Calling

$$g(x) = \left(-(b+x)^{\frac{1}{\rho}}\right) - \left(-(a+x)^{\frac{1}{\rho}}\right)$$

$$g'(x) = \frac{1}{\rho}\left((a+x)^{\frac{1}{\rho}-1} - (b+x)^{\frac{1}{\rho}-1}\right) < 0 \text{ because } \quad \frac{1}{\rho} - 1 > 0, \text{ and } \quad a+x < b+x.$$

Knowing that g(x) is a decreasing function, it will get its maximum value in x=0, in which case we will have

$$\left(-(b+x)^{\frac{1}{\rho}}\right)-\left(-(a+x)^{\frac{1}{\rho}}\right)<\left(-b^{\frac{1}{\rho}}\right)-\left(-a^{\frac{1}{\rho}}\right).\ \textbf{(Q.E.D.)}$$

Proof of Theorem 2:

The proof uses the method of complete or strong induction.

Firstly the result is shown to be true for a random variables V and U, this last being discrete such that $U \in \{0,1,\ldots,n\}$. Then applying the translation invariance and the positive homogeneity properties, it will also be proved for any discrete random variable $U \in \{kh,\ldots,(n+k)h\}$, (h>0), with h > 0. Finally, since any random variable can be arbitrarily closely approximated by a discrete random variable U with adequate h, k, the result will have been proven for any random variable.

Looking to (14) and reasoning by means of complete induction:

- If n=0 then U=0 and the result is trivial

- Assuming (14) to be true in the n-th case, let us examine the (n+1) case: Given (U, V) with $U \in \{0,1......n+1\}$ define (U*, V*) as $\left(U,V \middle| U>0\right)$.

Assuming $U^* \in \{1,\ldots,n+1\}$ the induction hypothesis states that:

$$H_\rho(U^*+V^*) \le H_\rho(U^*)+H_\rho(V^*).$$

Writing $\omega_0 = \Pr\left(U=0\right)$, $S_{V|0}(t)=\Pr\left(V>t \middle| U=0\right)$, for any $t>0$

$$\Pr\,(U>t)=\Pr\,\left(U^*>t \middle| U>0\right)\,\Pr\left(U>0\right),$$

This is the same than:

$$S_U(t)=(1-\omega_0)\ S_{U^*}(t).$$

On the other hand

$$\begin{aligned} S_V(t) &= \Pr\left(V>t\right)= \\ &= \Pr\left(V>t\middle|U=0\right)\Pr\left(U=0\right)+\Pr\left(V>t\middle|U>0\right)\Pr\left(U>0\right)= \\ &= S_{V|0}(t)\ \omega_0 + S_{V*}(t)\left(1-\omega_0\right). \end{aligned}$$

Also:

$$\begin{aligned} S_{U+V}(t) &= \Pr\left(U+V>t\right)= \\ &= \Pr\left(U+V>t\middle|U=0\right)\Pr\left(U=0\right)+\Pr\left(U+V>t\middle|U>0\right)\Pr\left(U>0\right)= \\ &= \omega_0\, S_{V|0}\left(t\right)+\left(1-\omega_0\right) S_{U*+V*}(t). \end{aligned}$$

Now according to Lemma 1:

$$\begin{aligned} &-S_{U+V}(t)^{1/\rho}-(-S_U(t)^{1/\rho})-(-S_V(t)^{1/\rho})= \\ &=-\left(\omega_0\, S_{V|0}+\left(1-\omega_0\right) S_{U*+V*}(t)\right)^{1/\rho}-\left(-\left(\left(1-\omega_0\right)S_{U*}(t)\right)^{1/\rho}\right)-\left(-\left(S_{V|0}(t)\ \omega_0+S_{V*}(t)\left(1-\omega_0\right)\right)^{1/\rho}\right)\le \\ &\le\left(1-\omega_0\right)^{1/\rho}\left(-S_{U*+V*}(t)^{1/\rho}-(-S_{U*}(t)^{1/\rho})-(-S_{V*}(t)^{1/\rho})\right)\le \\ &\le -S_{U*+V*}(t)^{1/\rho}-(-S_{U*}(t)^{1/\rho})-(-S_{V*}(t)^{1/\rho}). \end{aligned}$$

Next we reconstruct the corresponding expressions by adding one to each integral in both sides (the reader will check that in fact this changes nothing). Changing the variable to $v^t = z \Leftrightarrow t=\dfrac{\ln z}{\ln v}$, and integrating with respect to z we finally obtain:

$$\begin{aligned} &1-\int_0^1 S_{U+V}\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz-\left(1-\int_0^1 S_U\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz+1-\int_0^1 S_V\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz\right)\le \\ &\le 1-\int_0^1 S_{U*+V*}\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz-\left(1-\int_0^1 S_{U*}\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz+1-\int_0^1 S_{V*}\left(\frac{\ln z}{\ln v}\right)^{1/\rho}dz\right). \end{aligned} \tag{15}$$

The expression (14) can be written as

$$H_\rho(U+V)-\left(H_\rho(U)+H_\rho(V)\right)\le H_\rho(U^*+V^*)-\left(H_\rho(U^*)+H_\rho(V^*)\right)\le 0\,.$$

The last inequality is true by virtue of the induction hypothesis, so we conclude that the subadditivity axiom is also fulfilled by U and V. Therefore this premium calculation principle is a coherent risk measure. **(Q.E.D)**

Observation: Theorem 2 shows that the premium calculation principle for a whole life insurance contract with loss $Z = v^{T(x)}$

$$H_\rho(Z) = 1 - \int_0^1 \left(S_x\left(\frac{\ln z}{\ln v}\right)\right)^{1/\rho} dz, \quad 0 < \rho \leq 1 \ , z = v^t \ , \quad v = \frac{1}{1+i},$$

is a coherent risk measure. It cannot be considered as a generalization of Wang's result, but it theoretically supports the practice of modifying the instantaneous mortality rate in the case of death coverage products.

5. CALCULATION OF THE SINGLE PREMIUM FOR A LIFE INSURANCE WITH SURVIVOR'S COVERAGE (LIFE ANNUITY INSURANCE)

A continuous-time life annuity insurance (Bowers et al. (1997) p.134) is characterized by annuities payable continuously. Payments are made by the insurance company to an insured (x) as long as he is alive. In exchange the insured must pay the amount of the premiums to the company, either periodically or in the form of a single premium. In this case the risk can be modeled by the random variable T(x), residual life or time remaining until the insured's death. The following assumptions hold:

1. We have annuities payable continuously at the rate of 1m.u. per year (Bowers et al. (1997) p.134) until death.

2. i is the technical rate of interest.

3. Then the random variable T(x) has a distribution function $G_x(t)$ and a survival function $S_x(t)$ whose expressions depending on S(x) are given in (6).
4. Defining $v = (1+i)^{-1}$ the loss associated to the policy is then defined by means of the random variable the present value of payments $Z = \bar{a}_{\overline{T(x)}|}$, where:

$$\bar{a}_{\overline{t}|} = \int_0^t v^u du = \frac{1}{\ln v}(v^t - 1)$$

Applying the actuarial equivalence principle to get the pure premium (Bowers et al. (1997) p.135) we find:

$$\Pi(Z) = E(Z) = \int_0^{+\infty} v^t \, {}_t p_x \, dt,$$

where $v = \dfrac{1}{1+i} < 1$, and ${}_t p_x = S_x(t)$.

Changing the variable $v^t = z$ we obtain an expression for the pure prime based on $S_x(t)$

$$\Pi(Z) = \int_0^{+\infty} v^t \, S_x(t) \, dt = -\frac{1}{\ln v} \int_0^1 S_x\left(\frac{\ln z}{\ln v}\right) dz. \qquad (16)$$

The loaded premium using the distortion function "proportional transformation of the hazard function" given by (3) has the following form:

$$\Pi_\rho(Z) = -\frac{1}{\ln v} \int_0^1 S_x\left(\frac{\ln z}{\ln v}\right)^{\frac{1}{\rho}} dz \quad \text{with } \frac{1}{\rho} \le 1 \text{ , } \Pi_\rho \ge \Pi \;\; \Rightarrow \; \rho \ge 1. \qquad (17)$$

Now for the loaded premium to be greater or equal than the pure premium, the exponent must be less or equal than 1, so $\rho \geq 1$. Therefore these distorted probabilities satisfy the properties of a coherent risk measure (Wang (1996)).

For each value of $t = \dfrac{\ln z}{\ln v}$ given in (17) the distorted survival function is greater than the initial survival function, which means that the insured is deemed to have a lower risk of death. In this way, an annuity loss rate increases if the insured lives longer than expected. Thus the distortion function gives more weight to the tail of the residual time variable, resulting in a loaded premium.

Theorem 3.

The loaded premium (17) coincides with the pure premium of another random variable, with the same survival model law but an instantaneous mortality rate proportional to the instantaneous rate of variable X, with a proportionality factor equal to $\frac{1}{\rho}$.

Proof of Theorem 3:

In fact, if in expression (17) we introduce the change of variable $z = v^t$, we obtain:

$$z = 0, \quad t = \infty$$
$$z = 1, \quad t = 0$$
$$dz = v^t \ln v \, dt$$
$$\Pi_\rho(Z) = -\frac{1}{\ln v}\int_\infty^0 \left(S_x(t)\right)^{1/\rho} v^t \ln v \, dt = \int_0^\infty v^t \left(S_x(t)\right)^{1/\rho} dt.$$

Calling $S_Y(t) = (S_x(t))^{\frac{1}{\rho}}$ the expression of the instantaneous rate of variable Y equals (5):

$$\mu_Y(t) = \frac{1}{\rho}\mu_X(t).$$

Thus the instantaneous rate of the new variable is proportional to the original variable. **(Q.E.D.)**

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