

RISK LEVEL UPPER BOUNDS WITH GENERAL RISK FUNCTIONS

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Abstract

In the last teen years many new risk functions have been introduced (coherent risk measures, expectation bounded risk measures, generalized deviations, etc.) and many actuarial and/or financial problems have been revisited by using them. The use of new risk functions is well justified by the rapid development and evolution of the financial markets and the growing presence of skewness and kurtosis, among many other reasons, but the practical final result of many problems may critically depend on the concrete risk function we are drawing on. This paper deals with optimization problems involving risk functions and proposes several risk level upper bounds that apply regardless of the considered function. In particular both capital requirements and usual central moments and dispersions are bounded from above. The methodology is general enough and applies for perfect or imperfect financial markets, static or dynamic models, pricing or hedging issues, portfolio choice problems, optimal reinsurance problems, etc.

Key words. Risk measure, deviation, limit solution, risk level bounds.

A.M.S. Classification Subject, 90C48, 90C47, 90C34.

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Resumen

En los últimos diez años muchas nuevas funciones de riesgo han sido introducidas (medidas coherentes del riesgo, medidas de riesgo acotadas por la esperanza, desviaciones generalizadas, etc) y muchos problemas actuariales y/o financieros han sido nuevamente analizados bajo el prisma de éstas. El uso de nuevas funciones de riesgo está más que justificado por el rápido desarrollo y evolución de los mercados, y la cada vez mayor presencia de asimetrías y colas gruesas, entre otras muchas razones, pero el resultado final de muchos problemas de interés práctico puede depender de forma crítica de la función de riesgo considerada. Este artículo estudia problemas de optimización con funciones de riesgo, y obtiene cotas superiores del nivel óptimo del mismo, que acotan independientemente de la función de riesgo elegida. En particular, se acotan riesgos interpretables en términos de requerimientos de capital y otros que son dispersiones respecto a un momento central. La metodología es muy general, y es aplicable para mercados perfectos e imperfectos, estáticos o dinámicos, modelos de valoración y cobertura, temas de selección de inversiones, problemas de reaseguro óptimo, etc.

Palabras clave. Medidas de riesgo, desviación, solución límite, cota del nivel de riesgo.

Clasificación de la A.M.S., 90C48, 90C47, 90C34.

Clasificación del J.E.L., G11, G13, G22

I. Introduction

General risk functions are becoming more and more important in finance and insurance. Since the seminal paper of Artzner *et al.* (1999) introduced the axioms and properties of their “coherent measures of risk”, many authors have extended the discussion and the analysis. The recent development of new markets (insurance or weather linked derivatives, commodity derivatives, energy/electricity markets, etc.) and products (inflation-linked bonds, equity indexes annuities or unit-links, hedge funds, etc.), the necessity of managing new types of risk (credit risk, operational risk, etc.) and the (often legal) obligation of providing initial capital requirements have made it rather convenient to overcome the variance as the most important

risk measure and to introduce more general risk functions allowing us to address far more complex problems.⁴

Despite the growing interest in more general risk measurement methods there are no clear arguments justifying the use of a concrete risk function. Even for the standard Portfolio Choice Problem one can find different approaches using different risk measures. For instance, Benati (2003) minimizes the worst conditional expectation (WCE) in a static (or one period) framework. Also in a static setting, and using sample-linked finite probability spaces, Konno *et al.* (2005) minimize the absolute deviation and Mansini *et al.* (2007) minimize the conditional value at risk (CVaR) and compare with other measures. Alexander *et al.* (2006) compare the minimization of value at risk (VaR) and CVaR for a portfolio of derivatives. Anson *et al.* (2007) consider a vector optimization problem generated by several deviation measures reflecting the level of dispersion, skewness and kurtosis of a portfolio composed of hedge funds. Schied (2007) minimizes a general convex risk measure in a dynamic setting.

Many financial or insurance issues may lead to an optimization problem involving risk functions. References cited above are mainly related to portfolio choice theory but there are much more topics that may involve mathematical programming. So, pricing and optimal hedging in incomplete markets may imply the minimization of a risk measure among the differences between the pay-off to be priced (or hedged) and those pay-offs provided by the available self-financing hedging strategies (Föllmer and Leukert, 2000, Nakano, 2003, etc.). The loaded rate of equity indexed annuities (or unit-links), usual in the recent activity of many insurers, may be computed so as to control the issuer risk level (Barbarin and Devolder, 2005). Optimal reinsurance problems (Young, 1999, or Kaluszka, 2005), equilibrium pricing problems (Gao *et al.*, 2007), etc., may be also related to risk measures optimization.

Risk functions are almost never linear, though most of them are convex. Nevertheless, many authors have transformed the optimal risk problem so as to get a new equivalent linear problem. For example, Benati (2003) proposed a linear programming approach that permits us to deal with the WCE, Konno *et al.* (2005) showed that the minimization of the absolute deviation and the downside absolute semi-deviation may be also studied by linear programming methods and Mansini *et al.* (2007) extended their discussion so as to use linear programming when minimizing the CVaR and other risk

⁴ It may be worth to recall that the variance is not compatible with the Second Order Stochastic Dominance if asymmetric returns are involved in the analysis (Ogryczak and Ruszczyński, 2002).

functions. Besides, Balbás and Romera (2007) developed a linear programming analysis in infinite-dimensional Banach spaces and a simplex-like algorithm so as to hedge against the interest rate risk, and their study was extended in Balbás *et al.* (2009) in order to develop a general linear method applying for every risk minimization problem involving expectation bounded or deviation measures (Rockafellar *et al.*, 2006).

As said above we are far from a consensus about the “most appropriate” risk function to draw on, even when studying a classical problem. Furthermore, the practical result of many problems critically depends on the selected risk measure. For example, if we are computing initial reserves or capital requirements that a fund manager must incorporate, the choice of the risk function is far of being an irrelevant topic, and this situation also holds for more complex problems.

This paper deals with a general risk minimization problem and proposes several risk level upper bounds that apply independently of the considered risk function. In particular both capital requirements and usual moments (dispersions or deviations) are bounded from above. This seems to be an important question since it yields an objective reference that overcomes “conservative or risky selections” of the risk measurement procedure. The stability of the optimal strategy with respect to the chosen risk function will also be treated.

The paper’s outline is as follows. Second section will present the background, the general framework and the basic notations we will use throughout the article. Section III will present the risk level upper bounds that hold in a general risk minimization problem. The main idea is to apply those findings of Balbás *et al.* (2009) and use their infinite-dimensional linear programming approach so as to construct linear optimization problems whose feasible sets contain those involved in every particular risk measure or deviation measure.⁵ The main result is Theorem 4, that bounds both capital requirements linked measures and deviations. Sections IV and V extend the discussion and yield new improvements of the bounds that apply under particular assumptions. Theorems 7 and 8 seem to be their more important findings. Section VI analyzes the stability of the solution of the optimization problem with regard to the utilized risk function, and Theorem 12 is its most important result. Section VII is devoted to present two illustrative actuarial and financial problems. In particular, we will deal with

⁵ See also Balbás (2007).

an optimal hedging problem, and an optimal reinsurance problem. The last section of the paper points out the most important conclusions.

II. Preliminaries and notations

Let us assume that $t = 0$ and $t = T$ represent the current and a future date respectively. Consider the probability space $(\Omega, \mathfrak{F}, \mu)$ composed of the set Ω (states of nature), the σ -algebra \mathfrak{F} (information available at $t = T$) and the probability measure μ . Let be $p \in [1, \infty)$ and suppose that the convex cone $Y \subset L^p(\Omega, \mathfrak{F}, \mu)$ contains a set of pay-offs reachable at T (maybe by using self-financing strategies), $L^p(\Omega, \mathfrak{F}, \mu)$ (henceforth L^p for short) denoting the usual space of \mathfrak{F} -measurable random variables y such that the expectation of $|y|^p$ is finite.

Denote by $q \in (1, \infty]$ the conjugate of p ($1/p + 1/q = 1$). It is well known that the Riesz Representation Theorem states that L^q is the dual space of L^p (Luenberger, 1969). In particular, every real valued linear and continuous function on L^p takes the form $L^p \ni y \rightarrow E(q^* y) \in \mathfrak{R}$, $q^* \in L^q$ being an arbitrary element that only depends on the linear function we are dealing with, and $E(-)$ denoting the mathematical expectation of any random variable.

Consider a general risk function

$$\rho : L^p \rightarrow \mathfrak{R}$$

and a finite number of linear constraints $E(yq_j) \leq b_j$, $j = 1, 2, \dots, m$, where $q_j \in L^q$ and $b_j \in \mathfrak{R}$ are arbitrary, $j = 1, 2, \dots, m$.

We will deal with the risk minimization problem

$$\left\{ \begin{array}{l} \text{Min} \quad \rho(y) \\ E(yq_j) \leq b_j, \quad j = 1, 2, \dots, m \\ y \in Y \end{array} \right. \quad (1)$$

The cone constraint $y \in Y$ and the linear constraints above will be related in practice to standard restrictions. For example, short-selling restrictions, minimum required expected returns, budget constraints, fix positions in a

group of securities or in a single one, etc. Section VI will be devoted to present examples and will illustrate this fact.

Consider the convex and $\sigma(L^q, L^p)$ -closed subset of L^q given by

$$\Delta_R = \{z \in L^q : 0 \leq z, E(z) = 1\}. \quad (2)$$

If ρ is a coherent (Artzner *et al.*, 1999) and expectation bounded (Rockafellar *et al.*, 2006) risk measure then Rockafellar *et al.* (2006) have stated the existence of $\Delta_\rho \subset \Delta_R$, a convex and $\sigma(L^q, L^p)$ -compact subset of L^q such that

$$\rho(y) = \text{Max}\{-E(yz) : z \in \Delta_\rho\}. \quad (3)$$

Consequently, following Balbás *et al.* (2009), it may be easily proved that Problem (1) is equivalent to Problem

$$\left\{ \begin{array}{l} \text{Min} \quad \theta \\ \theta + E(yz) \geq 0, \quad \forall z \in \Delta_\rho \\ E(yq_j) \leq b_j \quad j = 1, 2, \dots, m \\ \theta \in \mathfrak{R}, y \in Y \end{array} \right. \quad (4)$$

(θ, y) being the decision variable. More accurately, y solves (1) if and only if there exists $\theta \in \mathfrak{R}$ such that (θ, y) solves (4), in which case $\theta = \rho(y)$ holds.

On the other hand if ρ is a lower range dominated deviation measure then Rockafellar *et al.* (2006) have stated that $\rho - E$ is coherent and expectation bounded. Then, if we still represent by $\Delta_\rho \subset \Delta_R$ the convex and $\sigma(L^q, L^p)$ -compact subset of L^q such that (3) holds for $\rho - E$ rather than ρ , then Problem (1) is equivalent to

$$\left\{ \begin{array}{l} \text{Min} \quad \theta \\ \theta + E(y(z-1)) \geq 0, \quad \forall z \in \Delta_\rho \\ E(yq_j) \leq b_j \quad j = 1, 2, \dots, m \\ \theta \in \mathfrak{R}, y \in Y \end{array} \right. \quad (5)$$

As pointed out by Balbás *et al.* (2009), (4) and (5) are linear regardless of the properties of the risk function ρ . Since the first constraint of (4) or (5) is valued on the Banach space $C(\Delta_\rho)$ composed of the real valued and (weakly*) continuous functions on the compact space Δ_ρ , the dual problem decision variable must belong to $M(\Delta_\rho)$, Banach space of the inner-regular σ -additive measures on the Borel σ -algebra of $C(\Delta_\rho)$ (see Balbás *et al.*, 2009, for further details on all of these properties). Thus, if $P(\Delta_\rho)$ denotes the (convex and $\sigma(M(\Delta_\rho), C(\Delta_\rho))$ -compact) set of inner regular probability measures on the Borel σ -algebra of Δ_ρ , then the dual of (4) becomes

$$\left\{ \begin{array}{l} \text{Max} \quad - \sum_{j=1}^m b_j \lambda_j \\ \sum_{j=1}^m q_j \lambda_j - \int_{\Delta_\rho} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\Delta_\rho) \end{array} \right. \quad (6)$$

where (λ, ν) is the decision variable, \geq_Y denotes the order in L^q given by $z_1 \geq_Y z_2$ if and only if $E(yz_1) \geq E(yz_2)$ for every $y \in Y$, and

$$\int_{\Delta_\rho} E(-) d\nu$$

denotes the element in $z_0 \in L^q$ such that

$$E(yz_0) = \int_{\Delta_\rho} E(yz) d\nu$$

for every $y \in L^p$.⁶

⁶ Notice that $L^p \ni y \rightarrow \int_{\Delta_\rho} E(yz) d\nu \in \mathfrak{R}$ defines a continuous linear function, so the existence of z_0 follows from the Riesz Representation Theorem (see Balbás *et al.*, 2009, for further details).

Similarly, the dual of (5) is

$$\left\{ \begin{array}{l} \text{Max} \quad -\sum_{j=1}^m b_j \lambda_j \\ 1 + \sum_{j=1}^m q_j \lambda_j - \int_{\Delta_\rho} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\Delta_\rho) \end{array} \right. \quad (7)$$

where the notations are analogous.

Since we may be dealing with infinitely many dimensions the absence of the so called duality gap between (4) and (6) (or (5) and (7)) is not guaranteed (Luenberger, 1969 or Anderson and Nash, 1987), *i.e.*, the optimal value of both problems may be distinct. To prevent this pathological possibility hereafter we will impose:

Assumption 1. The Slater Qualification holds, *i.e.*, there exists $y \in Y$ such that $E(yq_j) < b_j$, $j = 1, 2, \dots, m$.⁷

III. Capital requirements and deviations upper bounds

This section will provide upper bounds for the optimal value of (1) that will not depend on ρ . Problems (8) and (9) below will play a crucial role

$$\left\{ \begin{array}{l} \text{Max} \quad -\sum_{j=1}^m b_j \lambda_j \\ \sum_{j=1}^m q_j \lambda_j - \int_{\Delta_R} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\Delta_R) \end{array} \right. \quad (8)$$

⁷ If ρ reflects (maybe legal) capital requirements it hardly makes sense to assume that the infimum value of (1) or (4) might equal $-\infty$. Furthermore, if the primal problem is bounded, the Slater Qualification guarantees the absence of duality gap between (4) and (6) (Luenberger, 1969). Similar arguments also apply for Problems (5) and (7) (in particular, it is obvious that the optimal value of (5) cannot be $-\infty$).

and

$$\left\{ \begin{array}{l} \text{Max} \quad -\sum_{j=1}^m b_j \lambda_j \\ 1 + \sum_{j=1}^m q_j \lambda_j - \int_{\Delta_R} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\Delta_R) \end{array} \right. \quad (9)$$

Notice that Problems (6) and (7) and Problems (8) and (9) are almost similar. The only difference is in the first and last restrictions since (8) and (9) focus on the whole set Δ_R of (2). Since Δ_R is not necessarily $\sigma(L^q, L^p)$ -compact the convergence of the integral in the second constraint of (8) or (9) is not guaranteed. Thus, to prevent this caveat we will impose the decision variable ν to have $\sigma(L^q, L^p)$ -compact support (Luenberger, 1969).

We will denote by M and D the optimal values of (8) and (9) respectively. Notice that M and D do not depend on ρ , since they are given by the convex cone Y and the sets $\{b_j, j = 1, 2, \dots, m\}$ and $\{q_j, j = 1, 2, \dots, m\}$. Equalities

$$M = \infty \quad (10)$$

or

$$D = \infty \quad (11)$$

might hold, although sufficient conditions to prevent them will be provided throughout the paper. On the other hand, if (8) ((9)) were infeasible we would accept the convention $M = -\infty$ ($D = -\infty$).⁸

Lemma 2. If Δ_R is $\sigma(L^q, L^p)$ -compact then $M < \infty$ and $D < \infty$.

Proof. It is easy to verify that the $\sigma(L^q, L^p)$ -compactness of Δ_R guarantees that

$$R : L^p \rightarrow \mathfrak{R}$$

⁸ Needless to say that $D = -\infty$ cannot hold. However, we will not use this property, so we will not prove it either. Besides, $M = -\infty$ hardly could make sense in practice, mainly if ρ reflects capital requirements.

given by

$$R(y) = \text{Max}\{-E(yz) : z \in \Delta_R\} \quad (12)$$

is a coherent and expectation bounded risk measure. Then, Assumption 1 guarantees that (4) is feasible if Δ_R substitutes Δ_ρ . Hence, the usual primal-dual relationships (Luenberger, 1969) guarantee that (8) is bounded from above ($M < \infty$), unless (4) is unbounded in which case $M = -\infty$. Inequality $D < \infty$ may be proved with similar arguments.

Remark 3. 3.1. The $\sigma(L^q, L^p)$ -compactness of Δ_R will often hold in very important particular situations. For instance, it is satisfied if Ω is a finite set, since then L^q has only finite dimensions and Δ_R becomes the obviously closed and bounded set

$$\Delta_R = \left\{ \left(z(\omega_s)_{s=1}^C \right) : z \geq 0, \sum_{s=1}^C \mu(\omega_s) z(\omega_s) = 1 \right\},$$

C denoting the cardinal of Ω and $\mu(\omega_s)$ denoting the probability of the event ω_s , $s = 1, 2, \dots, C$ (obviously, without loss of generality we can assume that $\mu(\omega_s) > 0$, $s = 1, 2, \dots, C$).

3.2. More generally, with the same arguments as above Δ_R is $\sigma(L^q, L^p)$ -compact if μ is purely atomic with a finite number of atoms.

3.3. To deal with a finite set Ω is in some sense frequent in Finance and Insurance. For instance, in Portfolio Choice Theory many authors usually consider a one-period model, and final pay-offs are estimated by using the (finite) probability space generated by a real data sample (Konno and Yamazaki, 1991, Konno *et al.*, 2005, Mansini *et al.*, 2007, amongst many others). Ω is also finite if one draws on a dynamic discrete-time (usually incomplete or imperfect) pricing model and prices or hedges by minimizing risk levels (see, for instance, Jouini and Kallal, 2001, or Nakano, 2003).

Next let us prove that the value of M (D) provides us with upper bounds in a Risk Minimization Problem. These upper bounds apply for both deviations

or capital requirement linked risk measures and they do not depend on the concrete risk function ρ we are using.

Theorem 4. 4.1. If ρ is a coherent and expectation bounded risk measure and $y_0 \in Y$ solves (1) then the inequality $\rho(y_0) \leq M$ holds.

4.2. If ρ is a lower range dominated deviation measure and $y_0 \in Y$ solves (1) then the inequality $\rho(y_0) \leq D$ holds.

Proof. Let us prove 4.1 since 4.2 is similar. Assumption (1) guarantees the existence of (λ^*, ν^*) , dual solution of (6) such that $\rho(y_0) = -\sum_{j=1}^m b_j \lambda_j^*$. If we still denote by ν^* the inner regular probability measure with compact support on Δ_R that equals ν^* on Δ_ρ and vanishes out of Δ_ρ , then (λ^*, ν^*) is also (8)-feasible. Consequently, the inequality $-\sum_{j=1}^m b_j \lambda_j^* \leq M$ must hold.

Remark 5. 5.1. Let us suppose that $p \geq 2$. The standard deviation σ_2 is the most popular deviation measure when dealing with risk minimization problems in a classical framework. Unfortunately, the standard deviation is not lower range dominated, and the previous result does not apply in general. However, as pointed out in Ogryczak and Ruszczyński (2002), σ_2 is not appropriate (it is not consistent with the Second Order Stochastic Dominance) unless we are facing symmetric distributions. If every $y \in Y$ is symmetric then $\sigma_2(y) = \sqrt{2}\sigma_2^-(y)$, $\sigma_2^-(y) = E\left(\left(E(y) - y\right)_+^2\right)^{1/2}$ denoting the downside standard semi-deviation of every $y \in L^2$. Thus, for symmetric pay-offs (or returns, or final wealth), if we solve (1) with $\rho = \sigma_2$ then we will have $\sigma_2(y_0) \leq \sqrt{2} D$.

5.2. More generally, let us consider the integer s such that $p \geq s$, the s -deviation

$$\sigma_s(y) = E\left(\left|E(y) - y\right|^s\right)^{1/s}$$

and the downside s -semi-deviation

$$\sigma_s^-(y) = E\left(\left(E(y) - y\right)_+^s\right)^{1/s}.$$

Since σ_s^- is lower range dominated, similar arguments as above permit us to show that $\sigma_s^-(y_0) \leq D$ always holds if $y_0 \in Y$ solves (1) with $\rho = \sigma_s^-$, and

$$\sigma_s(y_0) \leq (2)^{1/s} D \quad (13)$$

holds if $y_0 \in Y$ solves (1) with $\rho = \sigma_s$ and every $y \in Y$ is symmetric.

5.3. It is known that for $s = 1$ the absolute deviation and downside semi-deviation satisfy $\sigma_1(y) = 2\sigma_1^-(y)$ for every $y \in L^1$ (recall that the expression holds even for non-symmetric random variables). Thus, if $y_0 \in Y$ solves (1) with $\rho = \sigma_1$, then

$$\sigma_1(y_0) \leq 2 D$$

holds, despite σ_1 is not lower range dominated in general and y_0 does not have to be symmetric.

IV. Upper bound improvements

This section will be devoted to improve the upper bounds above under special assumptions. In particular, we will consider those cases for which $Y \subset L^\infty$ (regardless of the value of $p \in [1, \infty)$) or $p = q = 2$.⁹

Next we will provide a first result applying when one is dealing with a essentially bounded attainable wealth. Since $L^\infty \subset L^p$ the particular case $Y \subset L^\infty$ may appear in practice. It is worth to recall that the dual space of L^∞ , that we will denote by Σ , is composed of those finitely additive measures $z : \mathfrak{F} \rightarrow \mathfrak{R}$ that are μ -continuous (i.e., $\mu(A) = 0 \Rightarrow z(A) = 0, A \in \mathfrak{F}$) and have finite variation (Luenberger, 1969,

⁹ Many actuarial and financial practical problems may be studied in a L^∞ framework. See for instance De Waegenaere and Wakker (2001) or Castagnoli *et al.* (2004).

or Anderson and Nash, 1987). If $y \in L^\infty$ and $z \in \Sigma$ then $\langle z, y \rangle \in \mathfrak{R}$ will denote the “standard product” of y and z .

Lemma 6. The set

$$\tilde{\Delta}_R = \{z \in \Sigma : 0 \leq z, \langle z, 1 \rangle = 1\} \quad (14)$$

is convex and $\sigma(\Sigma, L^\infty)$ -compact.

Proof. The convexity of $\tilde{\Delta}_R$ may be easily proved, so let us see its compactness. Since it is clearly $\sigma(\Sigma, L^\infty)$ -closed the Alaoglu’s Theorem (Luenberger, 1969) shows that it is sufficient to prove that $\tilde{\Delta}_R$ is norm bounded. If $z \in \Sigma$, $z \geq 0$ and $\langle z, 1 \rangle = 1$ then we have that $-1 \leq \langle z, y \rangle \leq 1$ for every $y \in L^\infty$ in the unit ball of L^∞ (since $-1 \leq y \leq 1$). Then z obviously belongs to the unit ball of the space Σ .

Theorem 7. If $Y \subset L^\infty$ then $M < \infty$ and $D < \infty$.

Proof. We will only prove the inequality $M < \infty$ since the other one is similar. Define

$$\tilde{R}(y) = \text{Max}\{-\langle z, y \rangle : z \in \tilde{\Delta}_R\}$$

for every $y \in L^\infty$. The previous lemma guarantees the consistency of the above definition. Then, Problem

$$\left\{ \begin{array}{l} \text{Min} \quad \theta \\ \theta + \langle z, y \rangle \geq 0, \quad \forall z \in \tilde{\Delta}_R \\ E(yq_j) \leq b_j \quad j = 1, 2, \dots, m \\ \theta \in \mathfrak{R}, y \in Y \end{array} \right.$$

satisfies the Slater Qualification. Indeed, take the element $y_1 \in Y$ with $E(y_1 q_j) < b_j$, $j = 1, 2, \dots, m$ (see Assumption 1), and $\theta_1 \in \mathfrak{R}$ with

$$\theta_1 > \text{Max}\{-\langle z, y_1 \rangle : z \in \tilde{\Delta}_R\}$$

whose existence trivially follows from the previous lemma. Therefore, the usual duality theory in Banach spaces for convex problems (Luenberger, 1969) ensures that its dual problem

$$\left\{ \begin{array}{l} \text{Max} \quad -\sum_{j=1}^m b_j \lambda_j \\ \sum_{j=1}^m q_j \lambda_j - \int_{\Delta_R} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\tilde{\Delta}_R) \end{array} \right.$$

is bounded from above (if it is not feasible then its value will be $-\infty$). Here the second constraint means that

$$\sum_{j=1}^m \lambda_j E(q_j, y) - \int_{\Delta_R} \langle z, y \rangle d\nu(z) \geq 0$$

holds for every $y \in Y$. Since the trivial immersion $L^q \rightarrow \Sigma$ transforms Δ_R in a subset of $\tilde{\Delta}_R$ it is clear that every inner regular with $\sigma(L^q, L^p)$ -compact support probability measure ν on Δ_R may be extended to $\tilde{\Delta}_R$, *i.e.*, the feasible set of Problem (8) may be embedded in the feasible set of the problem above. Thus, Problem (8) is bounded.

Throughout the rest of this section we will assume that $p = q = 2$ and Y is included in a finite-dimensional subspace $L \subset L^2$. The second property frequently holds. For instance, if we deal with a static (one period) model and the reachable pay-offs are those generated by combinations of a finite set of available assets.

Let us denote by

$$\pi : L^2 \rightarrow L$$

the standard orthogonal projection. We will consider the set

$$\ddot{\Delta}_R = Cl(\pi(\Delta_R))$$

Cl denoting closure. Notice that the closure may be computed in the norm topology of L because this space has a finite dimension (Luenberger, 1969).

Note that

$$E(yz) = E(y\pi(z)) \quad (15)$$

for every $y \in Y \subset L$ and every $z \in L^2$ because $z - \pi(z) \in L^\perp$, orthogonal of L . Whence, following the notations of (12),

$$R(y) = \text{Sup}\{-E(yz): z \in \Delta_R\} = \text{Sup}\{-E(y\pi(z)): z \in \ddot{\Delta}_R\}. \quad (16)$$

We will denote by M^* and D^* the optimal values of (8) and (9) respectively if Δ_R is replaced by $\ddot{\Delta}_R$. Obviously, if we are under the assumptions of Lemma 2 and Remark 3 and therefore Δ_R is $\sigma(L^2, L^2)$ -compact then $\ddot{\Delta}_R$ will be compact too, but the converse does not necessarily holds.

Theorem 8. If $\ddot{\Delta}_R$ is compact then $M^* < \infty$ and $D^* < \infty$. Furthermore, if $y_0 \in Y$ solves (1) and ρ is a coherent and expectation bounded risk measure (respectively, a lower range dominated deviation measure) then the inequality

$$\rho(y_0) \leq M^*$$

(respectively $\rho(y_0) \leq D^*$) holds.¹⁰

Proof. Once again we will only prove that $M^* < \infty$ and $\rho(y_0) \leq M^*$. First of all note that (16) leads to

$$R(y) = \text{Sup}\{-E(yz): z \in \Delta_R\} = \text{Max}\{-E(y\pi(z)): z \in \ddot{\Delta}_R\}.$$

According to Assumption 1, there exists $y_1 \in Y$ with $E(q_j y_1) < b_j$, $j = 1, 2, \dots, m$ and the compactness of $\ddot{\Delta}_R$ guarantees the existence of $\theta_1 \in \mathfrak{R}$ with

¹⁰ An analogous to Remark 5 also applies here.

$$\theta_1 > \text{Max}\{-E(y_1\pi(z)): z \in \ddot{\Delta}_R\}.$$

Hence, Problem

$$\left\{ \begin{array}{l} \text{Min} \quad \theta \\ \theta + E(y\pi(z)) \geq 0, \quad \forall z \in \ddot{\Delta}_\rho \\ E(yq_j) \leq b_j \quad j = 1, 2, \dots, m \\ \theta \in \mathfrak{R}, y \in Y \end{array} \right.$$

satisfies the Slater Qualification and its dual achieves its optimal value $M^* < \infty$ (unless it is unfeasible in which case $M^* = -\infty$). Moreover (15) shows that (1) is equivalent to (4) once $\pi(\Delta_\rho)$ replaces Δ_ρ (notice that $\pi(\Delta_\rho) \subset \ddot{\Delta}_R$ is compact) and the Slater Qualification shows that its dual achieves the optimal value $\rho(y_0)$. Since every inner regular probability measure on $\pi(\Delta_\rho)$ may be obviously extended to $\ddot{\Delta}_R$ the inequality $\rho(y_0) \leq M^*$ becomes obvious.

As already said Y will be included in a finite-dimensional space L if we deal with a one period model and there is a finite set of available securities whose pay-offs are $\{y_1, y_2, \dots, y_n\}$. Suppose that the risk free asset is also available, *i.e.*, suppose that (almost surely) constant functions are in Y . Then (15) leads to

$$E(z) = E(\pi(z)) \tag{17}$$

for every $z \in L^2$. In particular

$$E(z) = 1 \tag{18}$$

for every $z \in \ddot{\Delta}_R$. Then we have:

Corollary 9. Suppose that L is the linear manifold generated by an orthogonal system $\{y_1, y_2, \dots, y_n\} \subset L_+^2$. If (almost surely) constant functions are in Y then $\ddot{\Delta}_R$ is compact and the previous theorem applies.

Proof. Since $\ddot{\Delta}_R$ is obviously closed we only must prove that it is bounded. All the norm topologies in L are equivalent (Riesz Theorem, see Luneberger, 1969), so it is sufficient to see that $\ddot{\Delta}_R$ is bounded in the L^1 -norm. Owing to (18) this property would be obvious if we were able to show that $\ddot{\Delta}_R$ is included in the positive cone L_+^2 , and more easily, it is sufficient to see that $\pi(\Delta_R) \subset L_+^2$. Since System $\{y_1, y_2, \dots, y_n\} \subset L_+^2$ is orthogonal we have that

$$\pi(z) = \sum_{j=1}^m \frac{E(zy_j)}{\|y_j\|_2} y_j$$

for every $z \in L^2$, so $\pi(z) \in L_+^2$ if $z \in L_+^2$ because all the terms in the expression above are in L_+^2 .

V. Convex constraints

As said in the introduction Balbás *et al.* (2009) have provided complementary slackness necessary and sufficient optimality conditions that apply for all the dual pairs of linear problems presented in this paper. Moreover they developed a simplex-like algorithm that applies for most of the dual problems.

On the other hand, in practice, the restrictions of Problem (1) will be usually related to minimum required expected returns, budget constraints, short-sales, etc. If the market reflects frictions then some of these constraints will give up being linear, though most of them will be still convex. Convex pricing rules in finance or insurance have been studied, for instance, in Wang (2000), De Waegenaere and Wakker (2001) or Hamada and Sherris (2003) (see also Castagnoli *et al.*, 2004). Thus, Problem (1) may be extended so as to get

$$\left\{ \begin{array}{l} \text{Min} \quad \rho(y) \\ G_j(y) \leq b_j, \quad j = 1, 2, \dots, m \\ y \in Y \end{array} \right.$$

$\{G_1, G_2, \dots, G_m\}$ being real valued and continuous convex functions on Y . It is straightforward to obtain the natural extensions of Problems (4) and (5). Thus, bearing in mind the Duality Theory for Convex Optimization Problems of Luenberger (1969), if every G_j is positively homogeneous,¹¹ then

$$\left\{ \begin{array}{l} \text{Max} \quad - \sum_{j=1}^m b_j \lambda_j \\ \sum_{j=1}^m \lambda_j G_j - \int_{\Delta_\rho} E(-) d\nu \geq_Y 0 \\ \lambda_j \geq 0 \quad j = 1, 2, \dots, m \\ (\lambda, \nu) \in \mathfrak{R}^m \times P(\Delta_\rho) \end{array} \right.$$

becomes the dual of (4), and a similar modification applies for (5) as well. Here the second constraint above means that

$$\sum \lambda_j G_j(y) - \int_{\Delta_\rho} E(yz) d\nu(z) \geq 0$$

holds for every $y \in Y$. The necessary and sufficient optimality conditions of Balbás *et al.* (2009) may be also extended to the present case (see also Balbás, 2007), though it will be more difficult to use the new version in practice. The simplex-like algorithm will not apply anymore, but alternative convex-linked algorithms could be used. Besides, many theoretical results stated in Sections III and IV may be generalized so as to cover the convex case.

VI. Stability of the optimal solution

Let us deal again with the linear problem. Another important topic is related to the stability of the solution $y_0 \in Y$ of (1) with regard to the risk function ρ .

Proposition 10. Suppose that $(\lambda, \nu) \in \mathfrak{R} \times P(\Delta_R)$ solves (8) (respectively, (9)). Then the solution $y_0 \in Y$ of (1) and the optimal risk value $\rho(y_0)$ will

¹¹ i.e., $G_j(\alpha y) = \alpha G_j(y)$, $j = 1, 2, \dots, m$, $\alpha \geq 0$, $y \in Y$.

be the same for every coherent and expectation bounded risk measure (respectively, lower range dominated deviation measure) ρ such that $Sp(\nu) \subset \Delta_\rho$, $Sp(\nu)$ denoting the support of ν .¹²

Proof. As usual, we will deal with the coherent and expectation bounded case. If $Sp(\nu) \subset \Delta_\rho$ then $(\lambda, \nu) \in \mathfrak{R} \times P(\Delta_\rho)$ obviously solves (6) and the absence of duality gap guarantees that

$$\rho(y_0) = -\sum_{j=1}^m b_j \lambda_j.$$

Furthermore, the complementary slackness conditions (Anderson and Nash, 1987 or Balbás *et al.*, 2009) between (4) and (6) prove that system

$$\begin{cases} \sum_{j=1}^m b_j \lambda_j = \int_{\Delta_R} E(y_0 z) d\nu(z) \\ \lambda_j [b_j - E(q_j y_0)] = 0, & j = 1, 2, \dots, m \\ y_0 \in Y \end{cases}$$

along with the restrictions of (1) characterize the solution of (1), and this whole system does not depend on ρ .

Remark 11. 11.1. Following Balbás *et al.* (2009) it may be proved that $Sp(\nu)$ is finite under quite general conditions. Moreover, if the set $Sp(\nu)$ is finite then we can modify ν so that $Sp(\nu)$ can become a singleton. Indeed, suppose that $Sp(\nu) = \{z_1, z_2, \dots, z_k\}$. Then we can consider that $\nu = \sum_{i=1}^k t_i \delta_{z_i}$, δ_{z_i} denoting the usual Dirac delta that concentrates the total mass on $\{z_i\}$, $i = 1, 2, \dots, k$ (i.e., $\delta_i(z_i) = 1$). Since ν is a probability measure we have that $t_i > 0$, $i = 1, 2, \dots, k$, and $\sum_{i=1}^k t_i = 1$. Since Δ_R (respectively, Δ_ρ) is convex

¹² See Luenberger (1969) for a complete definition of $Sp(\nu)$.

we obviously have that $z_0 = \sum_{i=1}^k t_i z_i \in \Delta_R$ (respectively, $z_0 = \sum_{i=1}^k t_i z_i \in \Delta_\rho$).

Furthermore,

$$\int_{\Delta} E(yz) d\delta_{z_0} = E(yz_0) = \sum_{i=1}^k t_i E(yz_i) = \int_{\Delta} E(yz) d\nu$$

holds for every $y \in L^p$, where $\Delta = \Delta_R$ (respectively, $\Delta = \Delta_\rho$). Now it is obvious that δ_{z_0} may play the role of ν in Problems (8) or (9) (respectively, (6) or (7)).

As a consequence of the analysis above the conditions of Proposition 10 may be more easily verified in practice. Indeed, we have:

Theorem 12. If (λ, δ_{z_0}) solves (8) (or (9)) then Problem (1) has the same solution and the same optimal value for every coherent and expectation bounded risk measure (or lower range dominated deviation) such that $z_0 \in \Delta_\rho$. Moreover, the solution y_0 of (1) is characterized by system

$$\begin{cases} \sum b_j \lambda_j = E(y_0, z_0) \\ \lambda_j [b_j - E(q_j y_0)] = 0, \quad j = 1, 2, \dots, m \\ y_0 \in Y \end{cases} \quad (19)$$

along with the constraints of (1) if ρ is a coherent and expectation bounded risk measure, whereas the system becomes

$$\begin{cases} \sum_{j=1}^m b_j \lambda_j = E(y_0(z_0 - 1)) \\ \lambda_j [b_j - E(q_j y_0)] = 0, \quad j = 1, 2, \dots, m \\ y_0 \in Y \end{cases} \quad (20)$$

if ρ is a lower range dominated deviation.

VII. Actuarial and financial examples

As already said, many financial and actuarial problems may be studied by minimizing general risk functions. Since many references have been given, in this section we will just present two illustrative examples, and the interested reader may consult the cited references. Our first example will illustrate an application in finance, whereas the second one will deal with a classical actuarial topic. Both problems will be almost similar to Problem (1), but not identical, so some minor modifications of the statements in this article should be implemented in order to deal with the two proposed optimization problems. However, since these modifications are straightforward, we will not address them.

It is known that in an incomplete financial market many new pay-offs cannot be replicated and, consequently, they cannot be priced with a perfect hedging. An alternative may be to fix p and L^p , containing the sub-space Y of attainable pay-offs, and the risk function $\rho : L^p \rightarrow \mathfrak{R}$ to be used. Then, if $g \notin Y$ is the new security to be priced, and the trader buys $y \in Y$ so as to protect the sale of g , he/she can sell g for P_g euros and then find the optimal hedging strategy by solving

$$\left\{ \begin{array}{l} \text{Min} \quad \rho(y - g) \\ E(yq) \leq P_g \\ y \in Y \end{array} \right.$$

where q is the Stochastic Discount Factor that applies to price those pay-offs belonging to Y , *i.e.*, $E(qy)$ is the market price of every reachable pay-off $y \in Y$. The existence of q is guaranteed if, as usual, we impose the absence of arbitrage in the market (see, for instance, Cochrane, 2001, for further details about the notion of Stochastic Discount Factor). Once the problem above is solved we have the optimal risk level according to the measure ρ , as well as the optimal hedging strategy $y \in Y$. Thus, if ρ is a coherent and expectation bounded risk measure, then the theory developed in this article permits us to know whether the computed values of the risk level and the hedging strategy are stable or sensitive when one modifies ρ .

As a final comment, let us remark that the optimal hedging problem above may be interesting in Actuarial Mathematics as well. For instance, in order to price equity indexed annuities (or unit-links), since these products are

always related to incomplete markets (even if the annuities are linked to a complete financial market, the global market is not complete due to the stochastic behavior of mortality and survival).

The optimal reinsurance problem is “classical” in Actuarial Mathematics. Recent approaches may be found in Young (1999) and Kaluszka (2005), among others.

In general, suppose that $y_0 \in L^p$ is the (random) total amount that an insurance company will pay within a planning period. Suppose also that a reinsurance contract is accepted in such a way that $y \in L^p$ and $y_0 - y \in L^p$ will be the (random) amounts paid by insurer and re-insurer, respectively. Suppose finally that insurer and re-insurer apply the Expected Value Principle, and take $k \geq 1$ the proportion of the Pure Premium that they use in order to price (if the proportions are different then it is easy to see that we may only consider that proportion used by the re-insurer). Then the final wealth of the insurer will be

$$kE(y) - y,$$

so, if $S > 0$ is the minimum required pure premium, then the insurer will choose y so as to solve

$$\left\{ \begin{array}{l} \text{Min } \rho(kE(y) - y) \\ y \leq y_0 \\ E(y) \geq S \\ y \geq 0 \end{array} \right.$$

ρ being the applied risk function. Once again the developed theory may clarify whether the optimal reinsurance y and the optimal risk level $\rho(kE(y) - y)$ are really sensitive with respect to ρ .

VIII. Conclusions

Many financial or insurance problems have been recently revisited by drawing on more general risk functions. Amongst them, one can consider usual pricing, hedging or portfolio choice issues, optimal reinsurance problems, the loaded rate of unit-links, etc.

Until now many different risk functions have been recently proposed and there are no arguments justifying that a concrete example may outperform the remaining ones. Besides, the result of a practical problem may critically depend on the risk function we are dealing with. So, for instance, if we are computing initial capital requirements that a fund manager must incorporate, the choice of the risk function is far of being an irrelevant topic.

This paper has yielded several risk level upper bounds that apply regardless of the considered risk function. The methodology is general enough and applies for perfect or imperfect markets, static or dynamic models, pricing or hedging issues, portfolio choice problems, etc. Mainly, the only requirement is that one is optimizing a risk function to address a financial/insurance topic.

The stability of the optimal strategy with respect to the chosen risk function has also been studied, and illustrative practical examples have been provided.

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