BAYESIAN AND CREDIBILITY ESTIMATION FOR THE CHAIN LADDER RESERVING METHOD

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Abstract: Gisler and Wuthrich [8] describe how to calculate reserve estimates by means of Credibility and Bayesian estimators based on the development factors from different lines of business. This approach allows combining individual and collective claims information to get better estimations of the unknown reserves.

In this paper we compare the reserves estimates and the mean square error of prediction from two different models: Credibility and Bayesian ones. The objective is to show how the reserve estimates of these models are similar to the classical chain ladder models under certain distributional assumptions. The work includes a way of implementing the Bayesian model using Markov Chain Monte Carlo methods with the programming tool WinBUGS [15].

Key Words: Bayesian Models, Chain-Ladder, Credibility Theory, Markov Chain Monte Carlo, Normal Family.

Introduction

The determination of claim reserves for the outstanding liabilities is one of the most important tasks that an actuary performs to preserve the financial solvency of an insurance company.

The usual way to reproduce estimates about the unknown claim amounts for future years has been the use of forecasting methods based on the historical information, contained in a run-off triangle structure.

In some cases, the lack of information about past claims can constitute an obstacle to determinate reliable reserves. For that reason, actuaries often consider on the one hand the market experience and on the other one the company´s own experience: collective and individual information in credibility terminology. In this way, it is possible to add more information about the corresponding line of business.

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Credibility models allow the individual experience to be combined with the collective by reproducing the Bühlman's model [3] for measuring the weight between the individual and collective claims information.

Bayesian models use the likelihood distribution of individual outstanding claims and include the prior information (collective) in a natural way. The advantage of the Bayesian model is that they allow more statistical information about the reserve estimates, and also enable us to obtain the complete predictive distribution of the possible outcomes, in order to study risk measures.

In this paper, we focus on Bayesian models to estimate the claim reserving amounts using the statistical package WinBUGS [15]. This package is usually used to reproduce estimates via Markov Chain Monte Carlo (MCMC) methods. The paper also includes the link between credibility and Bayesian approach to statistical reasoning and model estimation. In particular we want to prove that under certain distributional assumptions and using non-informative priors, the reserve estimations for Bayesian and Credibility model are similar.

The structure of this paper is as follows. The first section summarizes the traditional chain ladder method (CLM). The second section states the modeling assumptions of Mack [10] and introduces the way in which the credibility theory can be implemented by means of individual and collective development factors. The third section describes the Bayesian formulation. The fourth section includes a numerical application using WinBUGS [15]. The last section provides the comparison results among models and set up conclusions.

1- The Chain-Ladder Method

In the run-off triangle, each row represents an origin year i for $0 \le i \le I$ and the column represents the development year j for $0 \le j \le J$. $C_{i,j}$ denotes cumulative claims (either incurred or paid) with a delay of j years from the origin year i.

Usually, the data consist of a triangle where $I = J$. However, other shapes of claim data can be assumed. In particular, we assume that the data information have an irregular pentagon shape where $I > J$ as in Table (1). Thus, the data consist of known cumulative claims for $i + j \le I$ and

unknown cumulative claims for $i + j > I$. In this paper we add the index k , $0 \le k \le K$, which specifies each line of business.
The column sum of the observed cumulative claims is defined as

$$S_{j,k}^{[t]} = \sum_{i=0}^{t} C_{i,j,k}\,, \quad \text{for } 0 \le j \le J, 0 \le k \le K \tag{1}$$

Table 1. Loss Development Data Structure

Development Year

Origin Year

i / j	**0**	**1**	**…**	**j**	**…**	**J – 1**	**J**
0	$C_{0,0,k}$	$C_{0,1,k}$	$\cdots$	$C_{0,j,k}$	$\cdots$	$C_{0,J-1,k}$	$C_{0,J,k}$
1	$C_{1,0,k}$	$C_{1,1,k}$	$\cdots$	$C_{1,j,k}$	$\cdots$	$C_{1,J-1,k}$	$C_{1,J,k}$
⋮	⋮	⋮	$\cdots$	⋮	$\cdots$	⋮	⋮
i = J	$C_{J,0,k}$	$C_{J,1,k}$	$\cdots$	$C_{J,j,k}$	$\cdots$	$C_{J,J-1,k}$	$C_{J,J,k}$
i = J+1	$C_{J+1,0,k}$	$C_{J+1,1,k}$	$\cdots$	$C_{J+1,j,k}$	$\cdots$	$C_{J+1,J-1,k}$	
⋮	⋮	⋮	⋮	⋮			
I - 2	$C_{I-2,0,k}$	$C_{I-2,1,k}$	$C_{I-2,2,k}$				
I - 1	$C_{I-1,0,k}$	$C_{I-1,1,k}$					
I	$C_{I,0,k}$						

Using this notation, the standard chain-ladder, development factors can be calculated as

$$f_{j,k} = \sum_{i=0}^{I-j-1} C_{i,j+1,k} \Bigg/ \sum_{i=0}^{I-j-1} C_{i,j,k} = S_{j+1,k}^{[I-j-1]} \Big/ S_{j,k}^{[I-j-1]}\,, \quad \text{for } 0 \le j \le N-1 \tag{2}$$

The aim of the CLM is to complete the empty triangle on the lower right corner of the table with the help of the development factors. In this paper the claim amount for the rows $i \le J$ has fully development and therefore we apply the development factors to the latest amounts known for the rest of the rows $(i > J)$ to estimate the unknown claim amounts:

$$\hat{C}_{i,J,k}^{CLM} = C_{i,I-i,k} * \prod_{j=I-i}^{J-1} f_{j,k} \tag{3}$$

In this way, it is possible to estimate the ultimate cumulative $C_{i,J}^{CLM}$ and obtain the reserve estimate for each accident year i:

$$R_{i,k}^{CLM} = \hat{C}_{i,J,k}^{CLM} - C_{i,I-i,k} \tag{4}$$

Additionally, we can find the estimate of the total amount of outstanding claims as

$$R_{Total,k}^{CLM} = \sum_{i=J+1}^{I} \hat{C}_{i,J,k} - \sum_{i=J+1}^{I} C_{i,I-i,k} \tag{5}$$

IAppendix (A) shows the data (claim amounts) from different lines of business. The claims amounts were taken from Gisler and Wuthrich [8], and were used for a practical analysis between models.

2- Credibility Theory approach

Mack [10] investigated the stochastic nature of the CLM, assuming a distribution-free model and specifying the first two moments for the cumulative claims, based on the following weak assumptions:

A1) Independence for the random variables $C_{i,j}$ between different accident years i.

A2) Existence of unknown factor $f_j > 0$ and $\sigma_j^2 > 0$, such that

$$E\left[C_{i,j+1,k} \middle| C_{i,0,k}, \dots, C_{i,j,k}\right] = C_{i,j,k} f_{j,k} \tag{6}$$

$$Var\left[C_{i,j+1,k} \middle| C_{i,0,k}, \dots, C_{i,j,k}\right] = C_{i,j,k} \sigma_{j,k}^2 \tag{7}$$

It is useful to work with the individual development factors to incorporate the claims amounts of each line of business (individual risk), as

$$Y_{i,j,k} = \frac{C_{i,j+1,k}}{C_{i,j,k}} \tag{8}$$

Formula (8) allows including the individual run-off triangle information about each line of business.

The model includes the set $\boldsymbol{B}_{\boldsymbol{j,k}} = \left\{ C_{i,t,k} ; i+t<I, t \le j, 0 \le k \le K \right\}$ which represent the complete observed information for $i+j \le I$. In addition, we consider the random variable $\boldsymbol{F}$ which consists in the set of development factors from the chain-ladder method.

Under model assumptions (6) and (7) the first two moments for the individual risk can be rewritten as

$$E\left(Y_{i,j,k} \middle| \boldsymbol{F}, \boldsymbol{B}_{\boldsymbol{j,k}}\right) = F_{j,k} \tag{9}$$

$$Var\left(Y_{i,j,k} \middle| \boldsymbol{F}, \boldsymbol{B}_{\boldsymbol{j,k}}\right) = \frac{\sigma_j^2\left(F_{j,k}\right)}{C_{i,j,k}}, \tag{10}$$

In the same way, the collective risk can be defined with mean and variance

$$E\left(F_{j,k} \middle| \boldsymbol{F}, \boldsymbol{B}_{\boldsymbol{j,k}}\right) = F_j \tag{11}$$

$$Var\left(F_{j,k} \middle| \boldsymbol{F}, \boldsymbol{B}_{\boldsymbol{j,k}}\right) = \frac{\tau^2\left(F_j\right)}{S_{j,k}^{I-j-1}}, \tag{12}$$

where $F_{j,k} = S_{j+1,k}^{[I-j-1]} \big/ S_{j,k}^{[I-j-1]}$ represents the chain-ladder factor defined in formula (2).

Observe that the credibility approach only uses the two first moment assumptions for the individual and collective risk as in Mack´s model. However, the reserve distribution is not available in both models.

The aim of the Credibility Theory is to estimate the individual credibility factor $F_{j,k}^{Cred}$ for each line of business in accordance with the individual and collective risk information.

Gisler and Wuthrich [8] developed the credibility theory for the estimation of the IBNR reserves, assuming a credibility factor which is similar to

$$F_{j,k}^{Cred} = \alpha_{j,k} F_{j,k}^{Ind} + \left(1-\alpha_{j,k}\right) F_j^{Coll} \tag{13}$$

where

- $F_{j,k}^{Cred}$ is a weighted mean from the individual and collective development factors.

- $F_{j,k}^{Ind}$ is the individual development factor for each line of business k.

  $$F_{j,k}^{Ind} = \frac{S_{j+1,k}^{[I-j-1]}}{S_{j,k}^{[I-j-1]}} \qquad (14)$$

- F_j^{Coll} is the collective development factor for all the lines of business (prior knowledge).

  $$F_j^{Coll} = E\left(F_j\right) \qquad (15)$$

- $\alpha_{j,k}$ is a parameter used to weight the individual and collective development factors.

  $$\alpha_{j,k} = \frac{S_{j,k}^{[I-j-1]}}{S_{j,k}^{[I-j-1]} + \frac{\sigma_{j,k}^2}{\tau_j^2}} \qquad (16)$$

- $\sigma_{j,k}^2$ is the variance for the individual development factors.

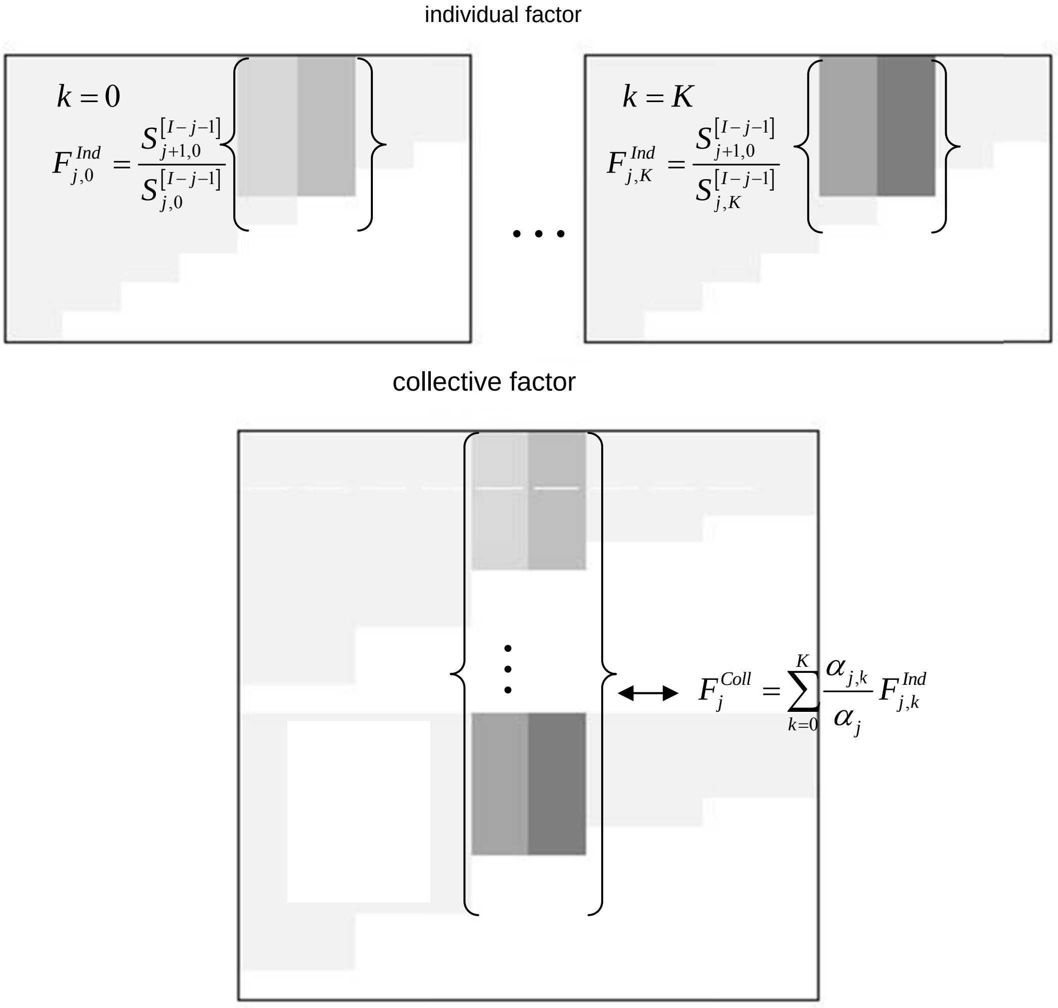
  $$\sigma_{j,k}^2 = E\left[\sigma_{j,k}^2\left(F_{j,k}\right)\right] \qquad (17)$$

- τ_j^2 is the variance for the collective factors

  $$\tau_j^2 = Var\left[F_j\right] \qquad (18)$$

The parameters $\alpha_{j,k}, \sigma_{j,k}^2$ and τ_j^2 can be estimated by using the standard estimators developed in Buhlmann and Gisler [4].
Diagram (A) shows the relation between the standard estimator for the individual $F_{j,k}^{Ind}$, collective F_j^{Coll} and credibility $F_{j,k}^{Cred}$ development factors.

Diagram A. Credibility Theory applied to IBNR reserves



The unknown claim amounts $C_{i,j,k}$ for $i+j>I$ are estimated using the development factors $F_{j,k}^{Cred}$ in the credibility claim estimates

$$C_{i,j,k}^{Cred}=C_{i,I-i,k}*\prod_{j=I-i}^{j-1}F_{j,k}^{Cred} \qquad (19)$$

Moreover, we can obtain the reserve estimate for each year i

$$R_{i,k}^{Cred}=\hat{C}_{i,J,k}^{Cred}-C_{i,I-i,k} \tag{20}$$

and its corresponding total reserve

$$R_{Total,k}^{Cred}=\sum_{i=J+1}^{I}\hat{C}_{i,J,k}^{Cred}-\sum_{i=J+1}^{I}C_{i,I-i,k} \tag{21}$$

Tables (2) and (3), summarize the estimated values for the credibility method. These results are similar to the numerical example results in Gisler and Wuthrich [8].

Table 2. Estimates of individual $\sigma_{j,k}^2$ and collective τ_j^2

j/k	$\sigma_{j,0}^2$	$\sigma_{j,1}^2$	$\sigma_{j,2}^2$	$\sigma_{j,3}^2$	$\sigma_{j,4}^2$	$\sigma_{j,5}^2$	τ_j^2
0	418.84	176.15	58.60	317.92	134.69	912.98	336.53
1	87.39	11.25	6.56	38.22	14.64	50.36	34.74
2	6.98	2.65	9.48	12.97	6.34	8.73	7.83
3	1.53	0.38	28.07	0.61	4.98	0.03	5.93
4	1.02	0.71	0.04	0.72	0.06	0.00	0.43
5	7.07	0.00	0.05	17.28	0.40	1.25	4.34
6	18.99	2.66	0.32	1.43	2.05	0.03	4.25
7	0.66	0.00	0.05	0.56	0.16	0.00	0.24
8	0.54	0.00	0.00	0.00	0.05	0.00	0.10
9	0.00	0.87	0.00	0.00	0.00	0.06	0.15

Table 3. Development factors (individual $F_{j,k}^{Ind}$, collective F_j^{Coll} and credibility $F_{j,k}^{Cred}$)

j/k	$F_{j,0}^{Ind}$	$F_{j,0}^{Cred}$	$F_{j,1}^{Ind}$	$F_{j,1}^{Cred}$	$F_{j,2}^{Ind}$	$F_{j,2}^{Cred}$	$F_{j,3}^{Ind}$	$F_{j,3}^{Cred}$	$F_{j,4}^{Ind}$	$F_{j,4}^{Cred}$	$F_{j,5}^{Ind}$	$F_{j,5}^{Cred}$	F_j^{Coll}
0	2.27	2.11	2.13	2.11	2.19	2.11	2.11	2.11	1.93	2.11	3.01	2.11	2.11
1	1.23	1.19	1.09	1.11	1.14	1.13	1.07	1.08	1.11	1.12	1.19	1.14	1.12
2	0.98	1.00	1.03	1.03	1.04	1.04	1.05	1.05	1.02	1.02	1.15	1.06	1.03
3	1.02	1.02	1.00	1.01	1.04	1.02	1.01	1.01	1.00	1.01	1.01	1.01	1.01
4	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
5	0.98	1.00	1.00	1.00	1.00	1.00	1.02	1.00	1.00	1.00	0.98	1.00	1.00
6	0.96	0.98	1.01	1.00	1.00	0.99	1.00	0.99	1.00	1.00	1.00	0.99	0.99
7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
8	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
9	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

3- Bayesian approach

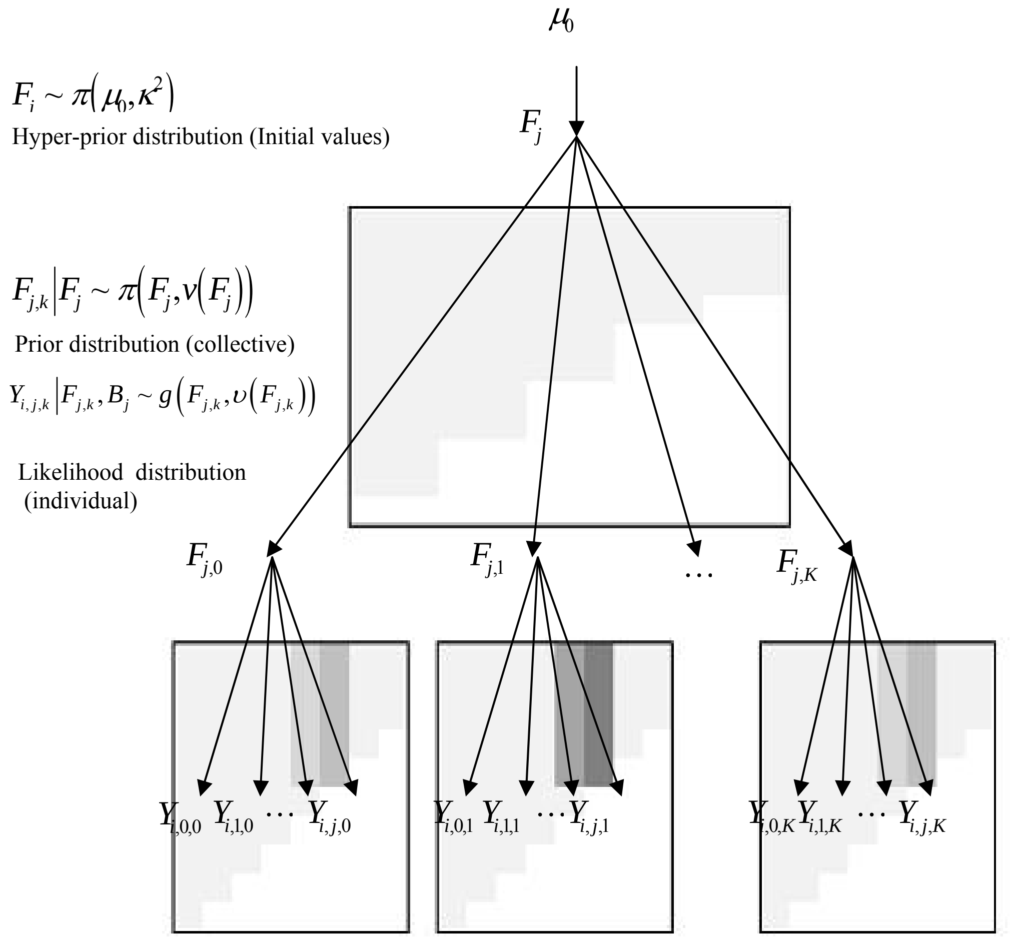
The relation between the credibility and Bayesian approaches were explained in Gisler and Wuthrich [8]. They replace $F_{j,k}^{Cred}$ by a Bayesian estimator $F_{j,k}^{Bayes}$ as

$$F^{Bayes} \approx \alpha_{j,k}\hat{F}_{j,k} + \left(1-\alpha_{j,k}\right)\hat{F}_j \qquad (22)$$

where $\hat{F}_{j,k}$ and $\hat{F}_j$ are the individual and the collective estimators respectively.

Diagram (B) shows the bayesian credibility structure: the likelihood distribution (individual risk), a prior distribution (collective risk) and a hyper-prior distribution to generate the initial values for the development factors $\hat{F}_j$.

Diagram B. Bayesian Credibility structure applied to IBNR reserves



The parameters $Y_{i,j,k}$, $F_{j,k}$ and F_j are defined as random variables and $\upsilon\left(F_{j,k}\right) = \sigma_j^2\left(F_{j,k}\right) \big/ C_{i,j,k}$, $v\left(F_j\right) = \tau^2\left(F_j\right) \big/ S_{j,k}^{I-j-1}$, μ_0 and κ^2 as known constant parameters.

The mean and variance of $Y_{i,j,k}$ (individual risk) are defined like in (9) and (10), as well as the mean and variance of $F_{j,k}$ (collective risk) is defined like in (11) and (12), respectively.

In Bayesian terminology, the likelihood function $g\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)$ describes how the random variables $Y_{i,j,k}$ are distributed given the random variable $F_{j,k}$ and the known set $\boldsymbol{B_{j,k}}$. On the other hand, the prior distribution $\pi\left(f_{j,k}\left|F_j\right.\right)$ describes the behavior about the individual development factors $F_{j,k}$ given the random variable F_j (collective factor). Finally, the hyper-prior distribution $\pi\left(f_j\right)$is used to generate the initial collective factors F_j.

In this way, conditionally, to $Y_{i,j,k}$ and $\boldsymbol{B_{j,k}}$, the posterior distribution of $\theta=\left(f_j, f_{j,k}\right)$is defined as:

$$\pi\left(f_j, f_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) = \frac{L\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)\pi\left(f_{j,k}\left|F_j\right.\right)\pi\left(f_j\right)}{\int L\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)\pi\left(f_{j,k}\left|F_j\right.\right)\pi\left(f_j\right)dF_j} \tag{23}$$

where $L\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)=\prod_{i=0}^{I}\prod_{j=0}^{I-j}\prod_{k=0}^{K}g\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)$

The Bayesian solution to the estimation of the individual development factor $f_{j,k}$ is given by the conditional mean $F_{j,k}^{Bayes}=E\left(F_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)$ defined as

$$\begin{aligned} E\left(F_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) &\propto \int f_{j,k}\int \pi\left(f_j, f_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)dF_j dF_{j,k} \\ &= \int f_{j,k}\pi\left(f_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)dF_{j,k} \end{aligned} \tag{24}$$

The link between the credibility and Bayesian factors lies when the Bayesian model works with conjugate distributions belonging to the exponential family distribution, in other words, if the likelihood function $g\left(y_{i,j,k}\left|F_{j,k}, \boldsymbol{B_{j,k}}\right.\right)$ and the prior distributions $\pi\left(f_{j,k}\left|F_j\right.\right)$and $\pi\left(f_j\right)$

belong to the exponential family. Then the posterior distribution $\pi\left(f_j, f_{j,k} \left| Y_{i,j,k}, \boldsymbol{B_{j,k}} \right.\right)$ belongs to the family of the natural conjugate priors. Buhlmann and Gisler [4] show that conjugate distributions from the Normal-Normal scheme result in a linear Bayesian factor, which is similar to the Credibility factor (13). Therefore, we used a hierarchical model, which is defined by distributions from the exponential family. In particular, we suppose a Normal distribution for the likelihood and prior distribution and a Log-normal for the hyper-prior distribution.

4- Numerical Bayesian application

Some applications of Bayesian models for outstanding reserve can be founded in Alba [1], England and Verrall [5] [6] and Ntzoufraz and Dellaportas [12].

Coded implementation of Bayesian models apply to IBNR reserves can be found in Alba [2], Scollnik [14], and Verrall [16].

The first stage for the implementation of our model consists in defining a likelihood function $g\left(y_{i,j,k} \left| F_{j,k}, \boldsymbol{B_{j,k}} \right.\right)$ to describe the known development factors $Y_{i,j,k}$ (upper left corner of the table) for $\boldsymbol{B_{j,k}} = \left\{ C_{i,t,k}; i+t<I, t \le j, 0 \le k \le K \right\}$.

For that sake we choose a Normal distribution with mean $F_{j,k}$ and variance $\upsilon\left(F_{j,k}\right) = \sigma_j^2\left(F_{j,k}\right) \big/ C_{i,j,k}$, where $Y_{i,j,k}$ is a random variable and $\upsilon\left(F_{j,k}\right)$ is a variance known and obtained among the individual $\sigma_{j,k}^2$ from Table (2). Observe that table (2) contains some values equal to cero. These values cannot be employed in the implementation of the BUGS code; therefore, we apply the following approximation

$$\sigma_{j,k}^2 = \begin{cases} 1/1000 & for \quad \sigma_{j,k}^2 = 0 \\ \sigma_{j,k}^2 & for \quad \sigma_{j,k}^2 \neq 0 \end{cases} \tag{25}$$

The second stage contain a prior distribution $\pi\left(f_{j,k} \left| F_j \right.\right)$ normally distributed with mean F_j and variance $v\left(F_j\right) = \tau^2\left(F_j\right) \big/ S_{j,k}^{I-j-1}$. Again we suppose that F_j is a random variable and $v\left(F_j\right)$ is a variance known from in Table (2).

From the credibility formula (13) we can observe that if $\tau_{j,k}^2$ grows larger, then $\alpha_{j,k} \to 1$.In other words, the credibility forecast equals the classical chain ladder forecast. To approximate this last situation, we may consider large variance setting

$$\tau_{j,k}^2 = 1000 \tag{26}$$

This is a non-informative prior density which reflects a total lack or ignorance of information.

Finally the third stage of the hierarchical model contains vague independent normal priors on Ψ_j used to generate the initial development factor f_j :

$$\Psi_j \sim dnorm\left(\mu_0, \kappa^2\right), \text{ with } \mu_o = 0, \quad \kappa^2 = 1000 \tag{27}$$

$$\log\left(F_j\right) = \Psi_j$$

The lognormal distribution guarantees initial positive values for the development factors f_j which guarantee the estimation of positive reserves. Summarizing, the model is defined in its the three levels by means of

$$\begin{aligned} &Y_{i,j,k}\left|F_{j,k}\right. \quad \sim \quad N\left(F_{j,k}, \upsilon_{i,j,k}^2\right) \\ &F_{j,k}\left|F_j\right. \quad \sim \quad N\left(F_j, \tau_{j,k}^2\right) \\ &\log\left(F_j\right) = \Psi_j \quad \sim \quad N\left(\mu_0, \kappa^2\right) \end{aligned} \tag{28}$$

where Ψ_j is an auxiliary variable used to generate a lognormal distribution for the initial development factors f_j .

We can predict the future development factors by means of the estimator factors $F_{j,k}^{Bayes}$ and the posterior predictive distribution $\pi\left(f_j, f_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)$. Unfortunately, the analysis of the marginal posterior distribution $\pi\left(f_j, f_{j,k}\left|Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)$ is not analytically tractable. However, we can obtain a numerical approximation by MCMC methods.

These methods include the use of the Gibbs sampler, which provides samples from the individual conditional posterior distribution of each parameter F_j and $F_{j,k}$.

$$\pi\left(f_j \left| F_{j,k}, Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) = \frac{\pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)}{\pi\left(f_{j,k} \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)} = \frac{\pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)}{\int \pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) dF_j}$$
$$\pi\left(f_{j,k} \left| F_j, Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) = \frac{\pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)}{\pi\left(f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)} = \frac{\pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)}{\int \pi\left(f_{j,k}, f_j \left| Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right) dF_{j,k}} \quad (29)$$

Considering a seed $\theta^{(0)} = \left(f_j^{(0)}, f_{j,k}^{(0)}\right)$, the first iteration of Gibbs sampling generates a sample $f_j^{(1)}$ from the individual posterior distribution $\pi\left(f_j \left| F_{j,k} = f_{j,k}^{(0)}, Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)$ and another sample $f_{j,k}^{(1)}$ from the individual posterior distribution $\pi\left(f_{j,k} \left| F_j = f_j^{(0)}, Y_{i,j,k}, \boldsymbol{B_{j,k}}\right.\right)$. As a result we also obtain the first iteration for the parameter. Then, each parameter is updated from its conditional distribution and then we finally fill the first iteration $\theta^{(1)} = \left(f_j^{(1)}, f_{j,k}^{(1)}\right)$. To fill the next iterations, for example for the iteration t, we need to update again the conditional distribution incorporating the values of the last iteration $\theta^{(t-1)} = \left(f_j^{(t-1)}, f_{j,k}^{(t-1)}\right)$. In order to incorporate the last iteration $t-1$, we can update the parameter $\theta^{(t)} = \left(f_j^{(t)}, f_{j,k}^{(t)}\right)$, successively. More details about the Gibb sampling algorithm can be found in Gamerman [7].
The model implementation in the computing package WinBUGS [15] is coded in Sanchez [13].

The Bayesian mean squared error of prediction (MSE) measures the variability of the reserves estimations $R_{i,k}^{Bayes}$

$$MSE\left(R_i^{Bayes}\right) = E\left[\left(C_{i,J} - E\left(C_{i,J}\right)\right)^2 \left| \boldsymbol{C}\right.\right] = Var\left(C_{i,J} \left| \boldsymbol{C}\right.\right) \quad (30)$$

The difference between the Mack´s [10] and Bayesian approach (30) is that the first one includes two parts $Var\left(C_{i,J}\left|\boldsymbol{C}\right.\right)$ and $\left(E\left(C_{i,J}\left|\boldsymbol{C}\right.\right)-\hat{C}_{i,J}\right)^2$ to calculate the MSE, and the second ones represents all the uncertainty only by means of $Var\left(C_{i,J}\left|\boldsymbol{C}\right.\right)$, which contains the uncertain parameters from the posterior distribution of all parameters.

WinBUGS yields the estimated factors $f_{j,k}^{Bayes}$ which include the MSE variability. Subsequently, these estimator factors can be used to estimate the mean and variance of the unknown cumulative variates $y_{i,j,k}$.

This way, it is finally possible to obtain directly the reserve jointly with the predictive distribution of the outstanding claims.

Thus, the estimations of the unknown claim amounts $C_{i,J,k}$ for the rows $(i > J)$ are given by the development factors $f_{j,k}^{Bayes}$:

$$\hat{C}_{i,j,k}^{Bayes} = C_{i,I-i,k} * \prod_{j=I-i}^{J-1} f_{j,k}^{Bayes} \text{, for } i > J,\ j > I-i,\ 0 \le k \le K \tag{31}$$

Now, it is possible to obtain the reserve estimate for each year i

$$R_{i,k}^{Bayes} = \hat{C}_{i,J,k}^{Bayes} - C_{i,I-i,k} \text{, for } i > J \tag{32}$$

as well as the corresponding total reserve

$$R_{Total,k}^{Bayes} = \sum_{i=J+1}^{I} \hat{C}_{i,J,k}^{Bayes} - \sum_{i=J+1}^{I} C_{i,I-i,k} \text{, for } i > J \tag{33}$$

Tables (4) and (5) show the development factor estimates for $F_{j,k}^{Bayes}$ and the reserves R^{Bayes} together with their prediction errors. An initial burn-in sample of 10,000 iterations was used. The results of these observations were discarded, to remove any effect from the initial conditions and allow the simulations to converge. Then further 50,000 simulations for each distributional assumption was run to reach the final results.

Table 4. Development factors for $F_{j,k}^{Bayes}$

Development year j

Development factor for each line of business k	0	1	2	3	4	5	6	7	8	9
$F_{j,0}^{Bayes}$	2.268	1.233	0.982	1.025	1.011	0.981	0.963	1.003	0.996	1.000
$F_{j,1}^{Bayes}$	2.134	1.094	1.032	1.002	0.998	1.000	1.014	0.999	1.000	0.990
$F_{j,2}^{Bayes}$	2.189	1.138	1.037	1.042	1.003	1.000	0.999	1.002	1.000	1.000
$F_{j,3}^{Bayes}$	2.108	1.070	1.054	1.013	1.004	1.014	0.996	0.995	1.000	1.000
$F_{j,4}^{Bayes}$	1.931	1.114	1.018	0.995	1.002	0.997	0.999	0.997	1.002	1.000
$F_{j,5}^{Bayes}$	2.997	1.191	1.147	1.006	1.000	0.979	0.996	1.000	1.000	1.004

Table 5. Reserves and prediction errors for the Bayesian model

Origin year i	*Business k = 0*		*Business k = 1*		*Business k = 2*		*Business k = 3*		*Business k = 4*		*Business k = 5*	
i	$R_{i,0}^{Bayes}$	*s.d*	$R_{i,1}^{Bayes}$	*s.d*	$R_{i,2}^{Bayes}$	*s.d*	$R_{i,3}^{Bayes}$	*s.d*	$R_{i,4}^{Bayes}$	*s.d*	$R_{i,5}^{Bayes}$	*s.d*
11	0.00	1.01	-8.98	29.54	0.00	1.25	0.00	2.95	0.00	1.50	0.62	3.00
12	-6.70	29.51	-10.52	31.76	0.01	1.93	-0.01	2.00	4.08	12.49	0.44	2.55
13	-5.36	68.41	-12.55	34.10	4.36	11.97	-34.32	70.02	-3.29	25.67	0.79	3.43
14	-25.05	116.00	3.31	59.46	0.90	20.67	-33.62	89.56	-3.49	63.11	0.00	4.28
15	-32.60	125.10	6.56	87.45	1.29	27.02	14.32	250.10	-4.83	53.29	-7.87	23.60
16	-24.80	123.10	1.37	74.11	4.11	24.41	28.59	270.80	-3.01	56.66	-5.10	18.87
17	-8.42	107.30	1.98	57.53	43.08	168.80	24.43	160.90	-14.08	119.40	-2.64	16.07
18	-16.55	120.20	14.20	56.23	72.32	189.80	138.50	256.40	8.48	113.30	18.12	38.17
19	76.48	256.30	54.87	92.91	144.10	185.10	225.30	352.80	155.50	199.40	16.56	692.70
20	528.80	645.30	187.10	399.20	431.90	233.40	652.00	559.80	354.80	286.30	8.49	1347.0
Σ	485.80	758.80	237.40	454.60	702.10	409.20	1015.0	865.40	494.20	412.00	29.40	1513.0

Conclusions

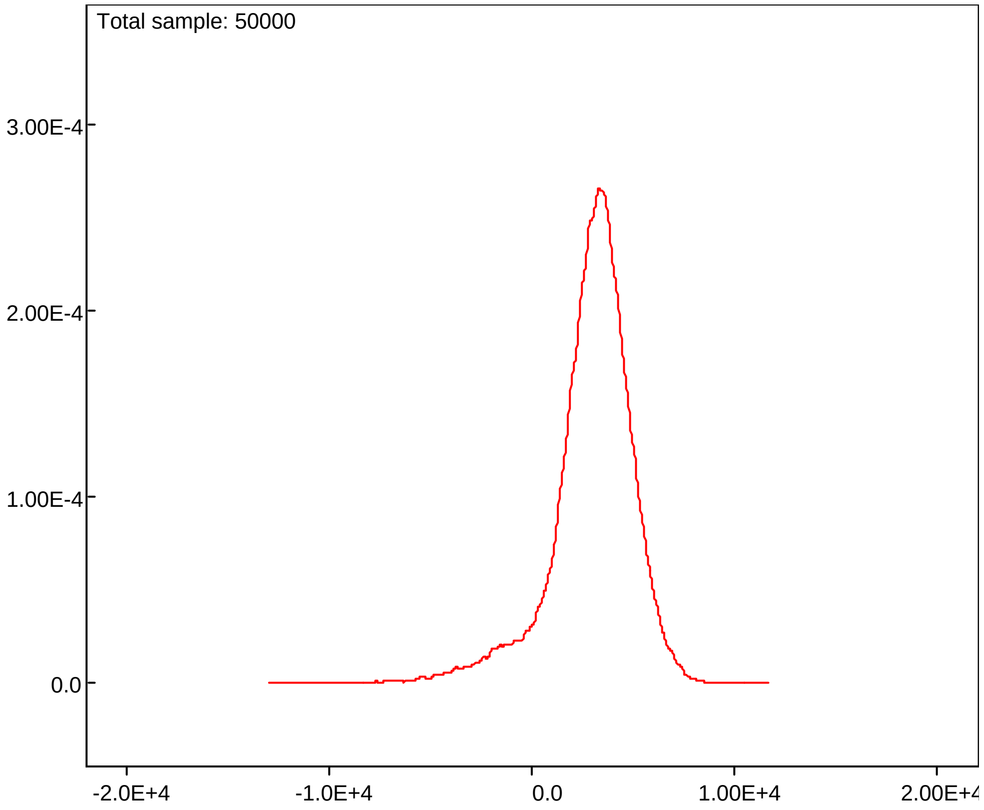
An important generalization in loss reserves modeling consists in considering more information furnished by different lines of business. Looking this way, we can develop a credibility formula which contains the CLM case when $\alpha=1$ and credibility mixtures otherwise. Moreover, the advantage of Bayesian approach is that we can obtain a full predictive distribution, rather than just the first and second moments as in Credibility and CL method. Plot (1) shows the posterior distribution for the collective risk and plot (2) the individual ones.

Table (6) shows on the one hand the reserves estimates when $\alpha = 1$ (non-informative prior); on the other, the MSE of prediction for each method. The results show how the use of non-informative priors in Bayesian analysis leads close reserves estimates as the MLE, when fitting the same model structure over the mean. For our example the Mack´s model has the smallest error predictor. However, this model does not express the idea of combining different lines of business as the Credibility and Bayesian models do. For both models the MSE of prediction are similar. Therefore, both models should be good to adjust the claim amounts. Finally, we can observe in the Bayesian model a small difference in the line of business 6. The reason is that there are small cumulative claims respects the other lines of business that affect the estimations of the reserves. To solve this problem we could remove the last line of business in order to make the same analysis. However the objective of this article was to compare the credibility and the Bayesian model with the same data information.

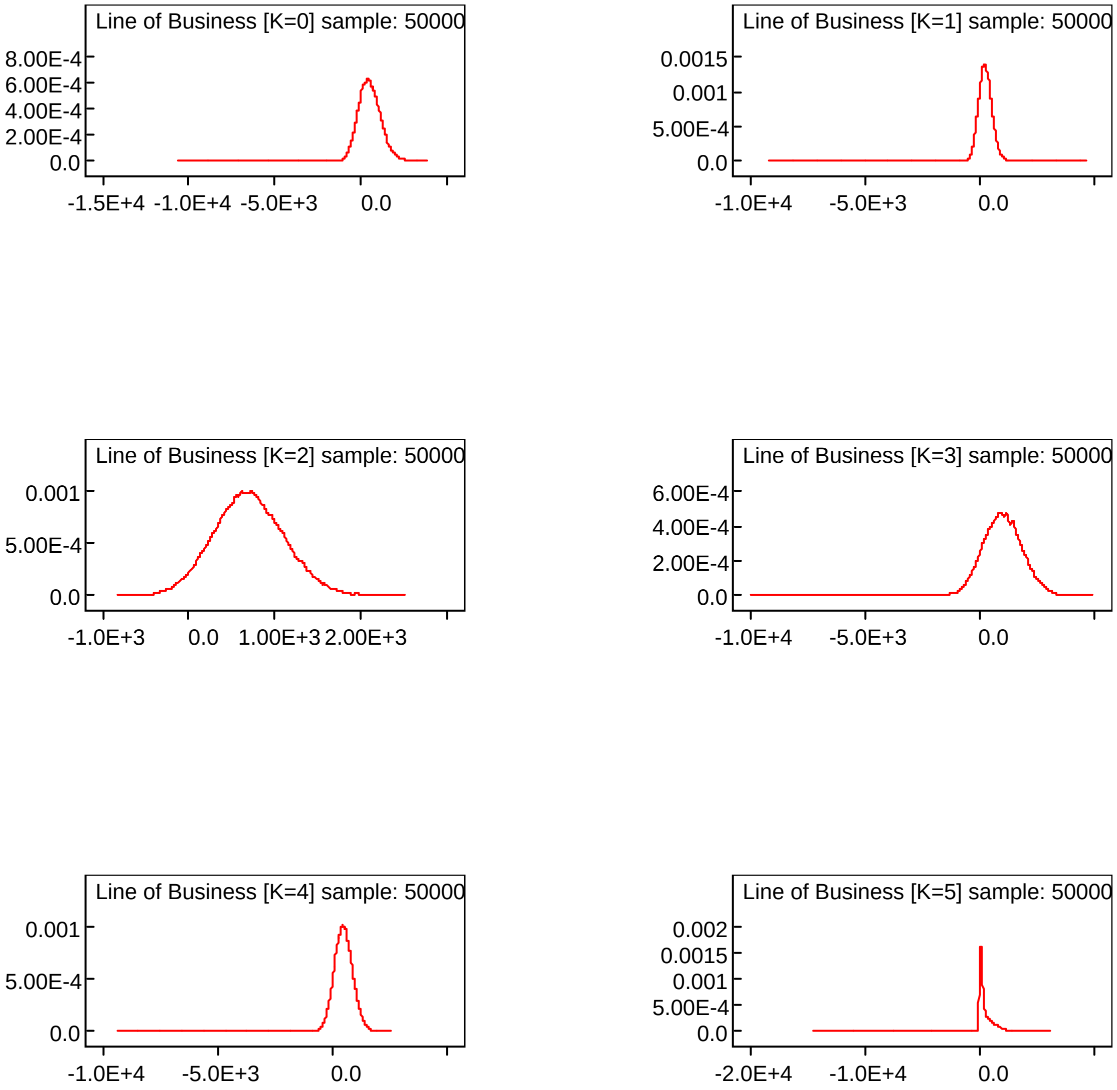
Table 6. Reserves for Credibility, Chain-ladder and Bayesian model

	t	Reserves			MSE	
k	Cred	MCL	Bayes	Cred	MCL	Bayes
0	504	486	486	498	510	759
1	244	235	237	402	424	455
2	517	701	702	520	565	409
3	899	1029	1,015	729	765	865
4	621	495	494	584	593	412
5	25	40	29	143	163	1513
R^{Total}	*2810*	*2987*	*2964*	*1254.2*	*1312.6*	*2049.0*

Plot 1. Predictive distribution for the collective reserve



Plot 2. Predictive distribution for the individual reserve



Appendix A. Cumulative claims from different lines of business.

	Triangle k = 0					**Development Year**						
	i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Origin Year	**0**	118	487	1232	1266	1266	1397	1397	1397	1492	1492	1492
	1	124	657	863	890	914	916	941	941	941	865	865
	2	556	2204	3494	2998	2983	3018	2458	2458	2470	2470	2470
	3	1646	2351	2492	2507	2612	2612	2608	1755	1755	1755	1755
	4	317	886	890	890	950	990	990	990	990	990	990
	5	242	919	1218	1224	1229	1249	1249	1249	1249	1249	1249
	6	203	612	622	639	667	647	647	647	647	647	647
	7	492	1405	1685	1668	1753	1742	1804	1804	1804	1804	1804
	8	321	1149	1728	1863	1877	1877	1877	1877	1877	1877	1877
	9	609	1109	1283	1294	1253	1255	1255	1255	1255	1255	1255
	10	492	1627	1622	1672	1672	1672	1672	1672	1621	1621	1621
	11	397	793	868	889	964	964	964	964	964	964	
	12	523	1098	1475	1489	1489	1489	1489	1489	1489		
	14	1786	2951	3370	3029	3211	3289	3325	3325			
	14	241	465	536	596	652	652	652				
	15	327	622	577	583	583	583					
	16	275	520	529	529	541						
	17	89	327	378	382							
	18	295	301	396								
	19	151	406									
	20	315										

	Triangle k = 1					**Development Year**						
	i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Origin Year	**0**	268	456	485	483	483	483	483	483	483	483	483
	1	268	520	577	579	579	579	579	579	579	579	579
	2	385	968	1017	1019	1019	1019	1019	1019	1019	1019	1019
	3	251	742	795	931	931	931	931	931	931	931	931
	4	456	905	1162	1164	1164	1164	1164	1164	1191	1191	1191
	5	477	1286	1376	1376	1373	1373	1373	1373	1373	1373	1373
	6	405	999	1172	1196	1196	1210	1210	1210	1210	1210	1210
	7	443	932	952	965	984	992	1012	1012	1012	1012	1012
	8	477	1046	1336	1362	1375	1375	1375	1375	1375	1375	1375
	9	581	1146	1316	1362	1391	1391	1391	1391	1391	1391	1391
	10	401	997	1229	1248	1281	1284	1264	1264	1264	1264	1264
	11	474	778	939	1321	1366	1392	1392	1392	1392	1392	

12	649	1420	1707	1709	1709	1709	1709	1638	1638
14	911	1935	2304	2307	2309	2309	2309	2362	
14	508	1054	1101	1071	1071	1071	1071		
15	389	790	868	909	1569	1569			
16	373	998	1091	1155	1201				
17	276	853	932	948					
18	465	820	859						
19	343	622							
20	254								

Triangle k = 2 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	92	442	541	541	528	528	528	528	528	528	528
1	451	1077	1085	1178	1212	1217	1217	1217	1217	1217	1217
2	404	717	834	849	849	850	850	850	850	850	850
3	203	572	813	875	878	910	912	1096	1089	1089	986
4	352	834	1048	1072	1088	1088	1088	1088	1088	1088	1088
5	504	1246	1272	1353	1285	1285	1285	1285	1285	1285	1285
6	509	1008	1061	1061	1061	1071	1071	1071	1071	1071	1071
7	229	580	630	670	672	672	672	672	672	672	672
8	324	815	871	859	867	777	777	777	777	777	777
9	508	805	906	969	971	971	971	971	971	971	971
10	354	641	833	842	842	842	842	842	842	842	842
11	431	847	854	915	918	918	918	918	918	918	
12	205	830	978	1034	1048	1048	1048	1048	1048		
14	522	1134	1064	1202	1202	1210	1210	1210			
14	567	925	915	957	953	953	953				
15	1238	1924	2034	1897	1897	1897					
16	355	1003	1137	1164	1196						
17	312	680	682	686							
18	246	352	418								
19	91	418									
20	130										

Triangle k = 3 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	330	1022	1066	1086	1094	1094	1094	1094	1094	1094	1094
1	327	873	1057	1076	1082	1082	1082	1082	1082	1082	1082
2	304	1137	1234	1460	1475	1588	1586	1586	1586	1586	1586
3	426	1289	1418	1574	1578	1634	2250	2044	2044	2044	2044
4	750	2158	2910	3071	3213	3199	3052	3052	3052	3052	3052
5	761	2164	2446	2570	2578	2558	2558	2558	2558	2558	2558
6	1119	2666	2946	3008	3021	3022	3019	3019	3019	3019	3019
7	917	2458	2892	3502	3629	3664	3887	3867	3697	3697	3697
8	905	2014	2459	2466	2554	2554	2554	2540	2540	2540	2540
9	1761	2990	3235	3795	3816	3841	3842	3860	3860	3860	3860
10	824	2063	2378	2368	2384	2368	2373	2373	2373	2373	2373
11	4364	6630	6850	6885	6923	6923	6923	6923	6923	6923	
12	493	1587	1780	1794	1838	1838	1838	1865	1865		
14	4092	7710	6596	7201	7292	7292	7292	7292			
14	1733	3647	3699	3780	3773	3773	3733				
15	1261	2658	3063	3036	3093	3095					
16	1517	3054	3335	3438	3438						
17	778	1212	1247	1215							
18	727	1661	1816								
19	561	1486									
20	459										

Triangle k = 4 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	486	964	1057	1106	1130	1130	1138	1131	1131	1131	1131
1	867	1669	1643	1717	1720	1724	1724	1724	1724	1724	1724
2	1285	1925	2204	2488	2507	2509	2510	2510	2436	2436	2436
3	395	994	1309	1442	1467	1467	1477	1477	1477	1477	1477
4	802	1468	1776	1823	1827	1832	1833	1833	1833	1833	1833
5	966	1967	2628	2743	2294	2338	2358	2358	2358	2358	2358
6	759	1766	1922	1863	1886	1886	1886	1886	1886	1886	1886
7	1136	2139	2219	1921	1931	1944	1947	1867	1867	1867	1867
8	1467	2243	2553	2598	2598	2598	2598	2598	2598	2598	2598
9	1309	2521	2660	2640	2639	2641	2659	2659	2659	2659	2659
10	877	2170	2341	2420	2516	2516	2431	2431	2431	2468	2468

11	1004	1963	2260	2226	2226	2215	2215	2059	2059	2059
12	1351	2579	2736	2759	2760	2766	2688	2737	2737	
14	906	2341	2667	2655	2655	2650	2650	2824		
14	563	1450	1575	1603	1654	1654	1675			
15	417	1006	1034	1049	1049	1050				
16	322	836	1046	1123	1143					
17	1047	1656	1689	1779						
18	497	843	877							
19	1021	1237								
20	302									

Triangle k = 5 **Development Year**

Origin Year

i / j	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	18	64	64	64	64	64	64	64	64	64	64
1	20	73	103	153	155	155	155	155	155	155	155
2	20	70	318	328	328	328	328	328	328	328	328
3	88	133	133	133	133	133	133	133	133	133	133
4	3	180	214	214	215	215	215	215	215	215	215
5	11	79	80	82	81	81	81	81	81	81	81
6	17	66	105	172	172	172	188	188	188	188	199
7	73	216	218	218	218	218	218	218	218	218	218
8	48	213	253	386	400	400	317	304	304	304	304
9	98	153	153	158	158	158	158	158	158	158	158
10	38	529	557	632	639	639	639	639	639	639	639
11	42	140	141	141	141	141	141	141	141	141	
12	64	95	95	102	102	102	102	102	102		
14	57	144	169	178	178	178	178	178			
14	85	178	188	186	186	186	186				
15	212	341	357	371	371	371					
16	56	152	187	246	246						
17	25	44	103	178							
18	19	137	140								
19	25	45									
20	7										

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